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THE THEORY AND DESIGN OF INDUCTANCE COILS

THE THEORY AND DESIGN
OF
INDUCTANCE COILS

BY

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WITH A FOREWORD BY

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FOREWORD

The subject of inductor design is one which I know has interested Dr. Welsby for many years and to which he has devoted much thought. It is of considerable importance to telecommunication engineers. Whereas inductance coils may be designed for many purposes, by " rough and ready " methods, such methods will not suffice when the coils are to be included in telephone transmission equipment, particularly as filter elements or in measuring apparatus. For such applications, the coils must be designed and constructed to give a more closely specified performance ; this is especially true with the modern methods of filter synthesis which are now coming into use.

To those whose task it is to design such coils, the information collected in this book should prove very valuable. Much of it may not be strictly new, but hitherto it had not been readily accessible in a form suitable for the coil designer, who is usually an engineer rather than a physicist. The author's personal contributions to the subject include much of Chapter IV and the work leading to the graphical methods of design described in Chapter X.

W. G. Radley

AUTHOR'S PREFACE

When the flow of electric current through a coil of wire is varied, the resulting change in the magnetic field surrounding the coil causes a voltage to be induced in the coil. The term " inductance coil " or " inductor " can be used to describe many practical applications of this principle, including, for example, the tuning coils used in radio apparatus, smoothing chokes for d.c. power supplies, the coupling and decoupling chokes used in valve amplifiers, control chokes for fluorescent lighting equipment, etc. The designer, requiring a coil to meet some particular specification, is faced with a bewildering array of possible types, from which he must select the one which is most suitable for the purpose, and then decide the correct size of wire and the number of turns to be used. In addition, if the coil is to have an iron core, the right material has to be chosen, and such details as lamination thickness, and air-gap length, fixed. It is to aid in the solution of problems of this nature that the present book has been prepared. Its object is to explain, in general terms, the underlying theory on which the design of inductance coils is based, and then to show how the theory can be simplified by the use of certain approximations and also to indicate to what extent these approximations are justified.

In this way it is possible to show how the performance of a given coil is influenced by its shape, size, core material, etc., and thus to show how its design should be changed to obtain any desired modification in its performance. The methods of designing various types of inductance coils are illustrated by numerical examples.

It should be pointed out that the terms " mumetal ", " radiometal ", etc., used to describe certain magnetic materials, are trade names which are used by courtesy of the manufacturers. The inclusion of curves showing some magnetic properties of these materials is intended mainly to illustrate theoretical points. Up-to-date values of permeability, loss coefficients, etc., obtainable with laminated, dust and ferrite cores, should be available from the manufacturers who supply the various materials.

In conclusion, the author would like to thank the Post Office Engineering Department for permission to use certain material which has been included in the book.

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CHAPTER I

INTRODUCTION

1. Impedance

When a steady current flows through a passive two-terminal device, (i.e., one which contains no internal source of energy), the voltage across it will settle down to a steady value, which is proportional to the current. The ratio between the voltage across the terminals to the current flowing is called the resistance of the device. If, instead of remaining constant, the current alternates with a constant amplitude, the voltage will also alternate, but will not necessarily be in phase with the current, or have the same waveform. If, for example, we are dealing with an inductance coil with an iron core, a large proportion of the total voltage across its terminals may be made up by the induced e.m.f. which is produced by the magnetic field of the coil. This induced e.m.f. will be out of phase with the current, and, as a result of hysteresis distortion, its waveform will not be exactly the same as that of the current. If the latter is a pure sine wave, the voltage waveform can be analysed into a fundamental component and a series of harmonic components. The ratio between the fundamental voltage across the coil and the current flowing through it is defined as the impedance of the coil. The behaviour of the latter can then be specified by stating the value of its impedance under any given conditions, and, at the same time, by expressing the harmonic distortion in terms of the relative amplitude of the induced e.m.f., at each harmonic frequency, compared with that of the fundamental current or voltage. The impedance of a coil cannot be completely represented by a single number, since it must express not only the ratio of the amplitudes of two alternating quantities but also the phase angle between them. The usual convention is to represent an impedance by a rotating vector, whose modulus gives the numerical ratio of voltage and current, and whose angular displacement from the zero axis gives the phase angle between current and voltage. The complex impedance is written as

$$Z = |Z| \angle \theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (1)$$

where $|Z|$ is the modulus and θ is the phase angle. For example, if the impedance of a coil is given as 1000 ohms with an angle of

60°, this means that an alternating current with an r.m.s. value of 1 milliampere would produce 1 volt r.m.s. across the coil, this voltage being 60° out of phase with the current.

2. The "j" notation

Instead of giving the modulus and angle, the impedance vector can be defined in another way. In Fig. 1, OA represents the vector and OB the zero axis. AB is a perpendicular from A on to the axis. It is apparent that the triangle OAB could be defined completely by stating the values of OA and AB. OA is the ratio between the components of voltage and current which are in phase with each other and it therefore denotes a pure resistance, which we will call R. AB is the ratio between the quadrature components and is defined as the reactance X. The impedance is then written as

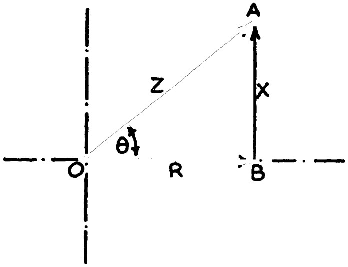

Fig. 1. Vector diagram representing the series components of the impedance of an inductance coil.

$$Z = R + jX \qquad (2)$$

where $j = \sqrt{-1}$.

The factor j is an operator which denotes the fact that the vector X is perpendicular to the zero axis.

3. Inductance

The induced e.m.f. in a coil is proportional to the rate of change of magnetic flux linking with the winding, and therefore, assuming no hysteresis distortion, it is proportional also to the rate of change of current in the winding. The constant of proportionality is called the inductance of the coil, and is measured in henries, one henry corresponding to an induced e.m.f. of 1 volt for a rate of change of current of 1 ampere per second. We can thus write

$$E = L \cdot \frac{dI}{dt} \text{ volts} \qquad (3)$$

where L is the inductance in henries.

If the current is varying sinusoidally, its value at time t will be given by an equation of the form

$$I = \hat{I} \cos \omega t \qquad (4)$$

where $\hat{I}$ is the maximum value of the current.

Then

$$E = L \, . \, \frac{d}{dt} (\hat{I} \cos \omega t) \qquad (5)$$

$$= \hat{I} L \, (-\omega \sin \omega t)$$

$$= \hat{I} \, \omega L \cos \left(\omega t + \frac{\pi}{2}\right)$$

$$= \hat{E} \cos \left(\omega t + \frac{\pi}{2}\right) \qquad (6)$$

and $$\frac{\hat{E}}{\hat{I}} = \omega L \qquad (7)$$

Thus the induced e.m.f. is also sinusoidal and has a phase angle 90° in advance of the current. An inductance coil has a type of impedance in which the voltage " leads " the current or, looked at in another way, in which the current " lags " behind the voltage. This type of impedance is said to be " inductive ", and the sign of its reactive component is taken as positive. The magnitude of the reactive component is clearly ωL, so that the impedance becomes

$$Z = R + j\omega L \qquad (8)$$

R and L are called the effective series resistance and effective series inductance, respectively. These terms will be explained later, and a more detailed definition of inductance will be given, referring to coils in which eddy-current losses and hysteresis distortion must be taken into account.

4. Definition of " Q "

Another way of defining the position and length of the impedance vector is to give the modulus of the vector and the tangent of the angle θ. In other words, the impedance is then defined in terms of the effective inductance L, the frequency f, and the value of the quantity $\tan \theta = \frac{\omega L}{R}$. The tangent of the phase angle is denoted by the symbol " Q ", and is called the " magnification " of the coil or simply the " Q " of the coil. This method is often used, particularly when dealing with coils for use in tuned circuits, since the inductance determines the resonance frequency of the tuned circuit and the sharpness of tuning depends on " Q ". Sometimes the term " dissipation factor " is used for the reciprocal of Q, whilst the " power *factor ", i.e., the value of cos θ, is sometimes used instead of Q.* The reader will recognize that all these terms are merely different ways of expressing the same thing, namely the phase angle of the coil impedance. The system used depends to some extent on the type of coil and the purpose for which it is to be used. For instance, the electronic engineer would probably use the value of Q when specifying a coil, while a power engineer, referring to a choke coil, would prefer to speak of its power factor, since he is interested in the watts dissipated in the coil rather than in its damping effect on the tuning of a resonant circuit.

CHAPTER II

EQUIVALENT CIRCUITS OF INDUCTANCE COILS

5. Series Circuit

The impedance of a coil can be expressed in the form

$$Z = R + j\omega L \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

The coil could therefore be replaced by an equivalent circuit, consisting of a non-reactive resistance R and an ideal inductance L, connected in series. The values of R and L will generally vary with frequency, and, in the case of iron-cored coils, will also depend on the amplitude of the alternating current and will be modified by the presence of a constant polarizing magnetic field, whether this is produced by a d.c. current in the winding or by some other means. The a.c. power dissipated in the coil is RI^2 watts, where I is the r.m.s. value of the current in the winding. At very low frequencies, the only source of power loss which matters is the d.c. resistance, and R will therefore be practically equal to R_{dc}. The inductance L will then be independent of frequency. As the frequency is raised, eddy-currents induced in the core, in the winding, and in any other conducting material in the vicinity of the coil, begin to take effect. Additional power is dissipated, and the value of R begins to rise with frequency, whilst the magnetic field produced by the eddy-currents will oppose the main field of the coil, causing the inductance to fall. At first, this fall in inductance is very slight, but, as soon as the eddy-current field becomes comparable in magnitude with the main field, the inductance falls more rapidly, finally becoming roughly proportional to the inverse square root of the frequency. In iron-cored coils, the effective series resistance is further increased by other losses in the magnetic core, and these, in turn, may also affect the total flux linking with the winding, and therefore change the value of L. In addition to the magnetic field, it must be remembered that there is an electric field, associated with the potential difference between the coil terminals, which may also have to be taken into account. The distributed capacitance between the turns of the winding acts as though an equivalent condenser were shunted across the coil. As the potential difference across the coil alternates, this condenser is successively charged and discharged, the resulting current modifying the

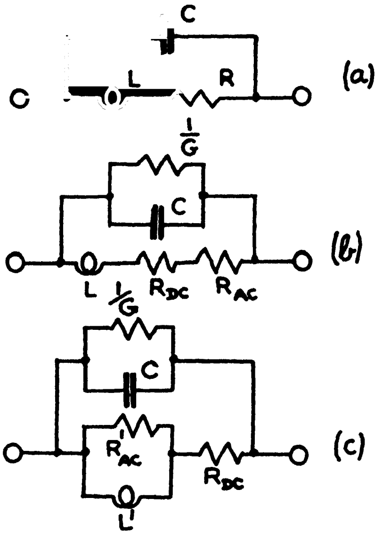

FIG. 2. Equivalent circuits of inductance coils.

amplitude and phase of the total current in the winding. Both R and L are therefore affected by the self-capacitance of the coil. Alternatively, we can replace the coil by an equivalent circuit in which R and L remain the same as they would be in the absence of self-capacitance, and then shunt a condenser C across this circuit, to represent the self-capacitance. The presence of the electric field causes power to be dissipated in the insulating materials used in the construction of the coil, particularly in the insulation of the wire. This "dielectric" loss should be negligible for air-spaced coils, or coils wound with wire which is insulated with low-loss materials, but in some cases, particularly if there is moisture present in the insulation, the dielectric loss may form a considerable proportion of the total power loss. The dielectric loss is best represented by a non-reactive conductance G (i.e., reciprocal of resistance), shunted across the self-capacitance C. We thus have the equivalent circuit shown in Fig. 2, in which R and L represent the behaviour of the coil when only the magnetic field is taken into account and C and G are added to account for the effects of the electric field. By careful design, it is generally possible to make C and G negligible, or, at any rate, to make G negligible, and to allow for C by suitable adjustment of the values of other components in the circuit in which the coil is to be used. In the following chapters, we shall first of all examine the theory of inductance coils, taking only the magnetic field into account, whilst the effects of self-capacitance and dielectric loss will be regarded as corrections which have to be applied where necessary.

6. Shunt Circuit

In a coil with a laminated core, it is sometimes found that most of the power loss takes place in the core, and the d.c. resistance is negligible compared with the total effective resistance. It will be shown later that, under certain conditions, it is convenient to represent the core power losses by a resistance shunted across an ideal inductance instead of being placed in series with it. These equivalent shunt values will be denoted by R′ and L′, respectively.

Then $Z = R + j\omega L$

$$\text{and } \frac{1}{Z} = \frac{1}{R'} + \frac{1}{j\omega L'} \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

rearranging equation (9),

$$Z = \frac{j\omega R'L'}{R' + j\omega L'}$$

$$= \frac{j\omega R'L' \; (R' - j\omega L')}{(R')^2 + (\omega L')^2}$$

$$= \frac{R' \; (\omega L')^2 + j\omega L'(R')^2}{(R')^2 + (\omega L')^2}$$

$$= \frac{R' + j\omega L' Q^2}{1 + Q^2}$$

$$\text{so that } R = \frac{R'}{1 + Q^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

$$\text{and } L = L' \left(\frac{Q^2}{1 + Q^2}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

It follows that, provided Q is fairly large, L and L′ are practically equal. In other words, the inductance of the coil will then be the same, whether it is expressed as a " series " or a " shunt " value. When Q is not large, care must be taken to distinguish between these two methods of expressing the impedance, and it must be remembered that a statement of the " inductance " of a coil may be misunderstood unless it is accompanied by the appropriate resistance value. In cases where doubt might arise over the exact definition, the difficulty can be avoided quite easily by stating the coil impedance as a modulus and angle instead of splitting it up into resistive and reactive components. For example, suppose the impedance of a coil, under some particular conditions, is given as 1000 ohms, with a phase angle of 60°. The Q is then tan $\underline{|60^\circ}$ and is equal to $\sqrt{3}$. Then, from (11),

$$L = L' \left(\frac{Q^2}{1 + Q^2}\right)$$

$$= \tfrac{3}{4} L'$$

There is a considerable difference here between the " series " and " shunt " inductance values, and care would have to be taken to state which one was meant when discussing the inductance of the coil.

7. D.C. resistance

If the d.c. resistance cannot be neglected, it can best be represented as a "series" value, so that, finally, we obtain the two complete alternative equivalent circuits shown in Fig. 2. The suffix "ac" has been added, in this diagram, to distinguish the d.c. resistance from the equivalent resistance, which expresses the losses due to the alternating magnetic and electric fields. Elsewhere in this book, unless otherwise stated, G and R_{dc} will be neglected and the suffix "ac" omitted, so that L and R, or L′ and R′, refer respectively to the components of the simple series and shunt equivalent circuits, which represent the impedance of the coil when self-capacitance, dielectric loss, and d.c. resistance are all neglected.

8. Effect of self-capacitance

Consider the equivalent circuits, Figs. 2 (*b*) and (*c*), in a case where R_{dc} and G are both negligible. The effect of the self-capacitance C on the shunt equivalent circuit will be to leave the shunt resistance unchanged, whilst the shunt reactance will be that of L′ and C in parallel. As a result, there will be an apparent increase in L′, the new value being L'_c, where

$$L'_c = \frac{L'}{1 - \omega^2 L'C} \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

$$\text{and } R'_c = R' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

The Q will be reduced to Q_c, where

$$Q_c = \frac{R'_c}{\omega L'_c} = \frac{R'}{\omega L'}(1 - \omega^2 L'C) \quad . \quad . \quad . \quad (14)$$

There will be a certain frequency, defined as the "self-resonant" frequency of the coil, for which $\omega^2 L'C = 1$ and the shunt reactance then becomes infinite. At this frequency, the impedance of the coil is a pure resistance equal to the corresponding value of R′.

The expressions for the impedance in terms of the "series" components L and R are rather more complicated. The impedance will be given by $(R + j\omega L)$ in parallel with $\frac{1}{j\omega C}$;

$$\text{i.e., } Z = \frac{\frac{1}{j\omega C}(R + j\omega L)}{R + j\omega L + \frac{1}{j\omega C}}$$

which, when rationalized, becomes

$$Z=\frac{[R(1-\omega^2LC)+\omega^2LCR]+j\omega[L(1-\omega^2LC)-CR^2]}{(1-\omega^2LC)^2+\omega^2C^2R^2}$$

Thus, if R_C and L_C are the new values of the series resistance and inductance,

$$R_C=\frac{R}{(1-\omega^2LC)^2+\omega^2C^2R^2} \quad . \quad . \quad . \quad . \quad (15)$$

$$L_C=\frac{L(1-\omega^2LC)-CR^2}{(1-\omega^2LC)^2+\omega^2C^2R^2} \quad . \quad . \quad . \quad . \quad (16)$$

$$\text{and } Q_C=\frac{\omega L_C}{R_C}=\frac{\omega L}{R}\left[(1-\omega^2LC)-\frac{CR^2}{L}\right]$$

$$=Q\left[1-\omega^2LC\left(1+\frac{1}{Q^2}\right)\right] \quad . \quad . \quad . \quad (17)$$

(14) and (17) should give the same value for Q_C, since this must be independent of the method used for expressing the impedance. This condition is clearly met, because (11) shows that $L'=L\left(1+\frac{1}{Q^2}\right)$.

The shunt expressions should be used for calculating the effects of self-capacitance in the general case, where L′, R′ and C may all be varying with frequency, since the series equations are too complicated to be of much use. When dealing with air-cored and dust-cored coils with relatively high Q values, the series equations can, however, be used to determine the corrections which must be applied to the series inductance and resistance in order to allow for self-capacitance.

(a) COILS WHOSE "TRUE" INDUCTANCE IS INDEPENDENT OF FREQUENCY

All the equations given above will still hold, even if the various component values of the equivalent circuits are themselves varying with frequency. Let us now consider a coil in which the magnetic field components produced by induced eddy-currents can be neglected. The induced e.m.f. produced by a given current in the winding will then be exactly proportional to frequency, so that the series inductance L will be independent of frequency. Let us write $\omega^2LC=x^2$ and $\frac{\omega L}{R}=Q$, so that (15) and (16) then become

$$R_C=\frac{R}{1-2x^2+\left(1+\frac{1}{Q^2}\right)x^4} \quad . \quad . \quad . \quad . \quad . \quad (18)$$

$$L_c = \frac{L\left[1-\left(1+\frac{1}{Q^2}\right)x^2\right]}{1-2x^2+\left(1+\frac{1}{Q^2}\right)x^4} \quad . \quad . \quad . \quad . \quad (19)$$

A further simplification can be made if the value of Q, at the frequency corresponding to x, is high enough for $\left(1+\frac{1}{Q^2}\right)$ to be taken as unity. (18) and (19) then reduce to

$$R_c = \frac{R}{(1-x^2)^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

$$L_c = \frac{L}{(1-x^2)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

At the same time, since $L' = L\left(1+\frac{1}{Q^2}\right)$ and we are assuming that $\left(1+\frac{1}{Q^2}\right)$ is unity, it follows that $L = L'$. (12) and (21) then become identical. Thus, when the inductance is independent of frequency and Q is high enough for $\frac{1}{Q^2}$ to be neglected in comparison with unity, the apparent inductance is increased by a factor $\frac{1}{1-x^2}$. The shunt resistance is unchanged, but the series resistance is increased by a factor $\frac{1}{(1-x^2)^2}$, so that the Q of the coil is reduced in the ratio $(1-x^2)$ to 1. This ratio is the correction factor, which must be applied to the " true " Q of the coil to allow for the effect of self-capacitance.

Under the above conditions, the self-resonant frequency will be given by

$$f_r = \frac{1}{2\pi\sqrt{LC}} = \frac{1}{2\pi\sqrt{L'C}} = \frac{f}{x}$$

$$\text{i.e., } x = \frac{f}{f_r}.$$

The self-capacitance correction factor can then be written as

$$Q_c = Q\left[1-\left(\frac{f}{f_r}\right)^2\right]$$

It should be noted that this equation will only hold if the " true " inductance is constant at all frequencies up to and including the self-resonant frequency.

(b) SUMMARY OF RESULTS

If the true magnetic inductance of the coil is constant for frequencies up to and including the self-resonant frequency, the effect of self-capacitance is to reduce the Q to Q_c, *where*

$$Q_c = Q\left[1-\left(\frac{f}{f_r}\right)^2\right] \quad . \quad . \quad . \quad . \quad . \quad (22)$$

If, in addition, the Q of the coil is sufficiently high, at both frequencies f and f_r, for $\left(1+\frac{1}{Q^2}\right)$ to be taken as unity the series and shunt inductances are equal, and given by

$$L_c = \frac{L}{1-\left(\frac{f}{f_r}\right)^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

The series resistance is then

$$R_c = \frac{R}{\left[1-\left(\frac{f}{f_r}\right)^2\right]^2} \quad . \quad . \quad . \quad . \quad . \quad (24)$$

and the shunt resistance is unchanged by the self-capacitance. If the above conditions are not met, it is better to work in terms of shunt values only. In this case, the shunt resistance is still the same as it would have been with no self-capacitance, whilst the shunt inductance is given by

$$L_c = \frac{L'}{1-\omega^2 L'C} \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

$$\text{also } Q_c = Q\,(1-\omega^2 L'C) \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

The values of L', C, Q, and Q_c are those at the frequency corresponding to ω.

(c) POWER LOSS DUE TO SELF-CAPACITANCE

It has been shown that the self-capacitance of a coil causes an apparent rise in its series resistance, and therefore an increased power loss for a given current flowing through the coil. This is quite independent of any dielectric loss associated with the self-capacitance, and will occur even if this is negligible. The explanation is that the alternating potential difference across the equivalent condenser causes a pulsating current to flow into and out of the condenser as it is successively charged and discharged. If the total current flowing in at one terminal and out at the other is kept constant in amplitude, *it follows that the presence of the condenser current must modify the current in the winding.* The phases of the various currents are such that the r.m.s. value of the current in the winding is increased, resulting in increased power losses. A pure capacitance, shunted across an iron-cored coil, therefore increases the " copper " losses, and also the flux density and the core losses, for a given total current through the coil. If, instead of keeping the current constant, we maintain a constant voltage across the coil terminals, the flux density and the power losses are unchanged by the addition of a shunt capacitance, as is shown by the fact that the apparent shunt resistance of the coil does not change. The condenser currents, however, which are 90° out of phase with the applied voltage, cause a change in the " wattless " current in the winding, and therefore a change in the shunt inductance of the coil.

The self-resonant effect is sometimes an advantage in choke coils, since it raises the effective impedance. Generally, however, it is an undesirable property, particularly if the coil is to be used in a series resonant circuit, because it reduces the Q of the coil and also causes its series inductance to vary with frequency. In a parallel tuned circuit, the self-capacitance is not so important, since it can be allowed for by readjustment of the tuning condenser and does not cause additional damping of the tuned circuit. Even so, trouble may be caused by the dielectric losses which accompany the self-capacitance, causing the latter to behave like a condenser of comparatively high power factor.

CHAPTER III

AIR-CORED COILS

Coils which have no magnetic material in their vicinity are said to be " air-cored " ; this term includes coils wound on non-magnetic formers as well as those in which the turns are actually air-spaced.

9. Inductance formulae

(a) SINGLE-LAYER SOLENOIDS

Consider a long, thin solenoidal coil consisting of a single layer of conducting strip, wound so that the insulating gap between successive turns is negligibly small. Such an arrangement will give a close approximation to an ideal " current sheet " in which the current density is uniform over the whole surface of the winding. The series inductance of such a coil will be given by

$$L=\frac{4\pi N^2 A}{l}\,.\,10^{-9}\text{ henries} \quad . \quad . \quad . \quad . \quad (27)$$

where N is the number of turns and l is the length of the coil in centimetres. If the cross-section is circular, with a radius r cm. (measured from the axis to the centre line of the conductor), the area A will be πr^2, and the inductance is then

$$L=\frac{4\pi^2 r^2 N^2}{l}\,10^{-9}\text{ henries} \quad . \quad . \quad . \quad . \quad (28)$$

As the coil is made shorter, the field inside it is no longer uniform, and the inductance becomes a function of the ratio of length to radius. (28) must then be multiplied by a correction factor K_n, whose value is

$$K_n=\frac{1}{1+0{\cdot}9\left(\frac{r}{l}\right)-0{\cdot}02\left(\frac{r}{l}\right)^2} \quad . \quad . \quad . \quad (29)$$

These formulae will also apply to coils wound with round wire instead of strip, provided the turns are fairly close together, but, if the turns are spaced well apart, a correction must be made for the fact that *the current is concentrated in the wires and not uniformly distributed over the coil. The " current-sheet " inductance, obtained from (29),* must be multiplied by

$$\left[1-\frac{l\,(\mathrm{A}+\mathrm{B})}{\pi r \mathrm{N} \mathrm{K_n}}\right] \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (30)$$

$$\text{where } \mathrm{A}=2{\cdot}3 \log_{10} 1{\cdot}73\,\frac{d}{c} \quad . \quad . \quad . \quad . \quad . \quad . \qquad (31)$$

$$\text{and } \mathrm{B}=0{\cdot}336\left(1-\frac{2{\cdot}5}{\mathrm{N}}+\frac{3{\cdot}8}{\mathrm{N}^2}\right) \quad . \quad . \quad . \quad . \quad . \qquad (32)$$

d is the diameter (in cm.) of the circular conductor, and c is the winding " pitch " ; i.e., the centre-to-centre distance between successive turns. The relative magnitudes of the various corrections in a typical case will be illustrated by an example.

Example

It is required to find the inductance of a solenoidal coil which has ten turns of solid round wire with a mean radius of 1 cm., if the length of the coil is 3 cm. The " Nagaoka constant ", $\mathrm{K_n}$, will be

$$\mathrm{K_n}=\frac{1}{1+0{\cdot}9\,(\frac{1}{3})-0{\cdot}02\,(\frac{1}{9})}=0{\cdot}768.$$

The inductance is then

$$\mathrm{L}=\frac{4\pi^2}{3}\times 0{\cdot}768\;.\;10^{-7}\text{ henries}$$

$$=1{\cdot}01\text{ microhenries.}$$

The next step is to work out the " current-sheet " correction, to see if this needs to be taken into account ; but, before this can be done, some value must be assumed for the wire diameter. Suppose this is 0·10 cm. The pitch is $\frac{3}{10}$ cm., so that the ratio $\frac{d}{c}=\frac{1}{3}$.

$$\text{Then } \mathrm{A}=2{\cdot}3 \log_{10} \frac{1{\cdot}73}{3}$$

$$= -0{\cdot}548$$

$$\text{and } \mathrm{B}=0{\cdot}336\,[1-0{\cdot}25+{\cdot}038)= +0{\cdot}266$$

so that $(\mathrm{A}+\mathrm{B}) = -0{\cdot}282$.

The correction factor is thus $\left(1+\frac{3\times 0{\cdot}282}{7{\cdot}68\pi}\right)$

$$=1{\cdot}035.$$

In this case the correction is only $3\frac{1}{2}\%$, and would probably be ignored. A coil of this type is so simple to construct that it is usually *necessary merely to obtain a rough estimate of the inductance, final adjustment then being made by trial and error.* The "current-sheet" correction would only become considerable for a coil wound with very thin wire, and with relatively wide spacing. For instance, if the ratio $\frac{d}{c}$ had been $\frac{1}{100}$ instead of $\frac{1}{3}$, we should have had

$$\begin{aligned} A &= 2{\cdot}3 \log_{10} 0{\cdot}0173 \\ &= -4{\cdot}05 \end{aligned}$$

and, for a ten-turn coil,

$$B = +0{\cdot}27$$

so that $(A+B) = -3{\cdot}78$.

The correction factor for a coil of the dimensions given above will then be $\left(1+\frac{3\times 3{\cdot}78}{7{\cdot}68\pi}\right) = 1{\cdot}47$.

A coil with the ratio $\frac{d}{c}$ as small as $\frac{1}{100}$ would however be very unlikely in practice and, for most purposes, the correction can be neglected.

(*b*) MULTILAYER SOLENOIDS

With multilayer coils, the spacing between the turns is usually sufficiently small for the inductance to be quite independent of the spacing, and is determined only by the number of turns and the overall dimensions of the winding. The equivalent of the Nagaoka constant K_n is now a function, not only of the length and mean radius of the coil, but also of the depth of the winding t (*see* Fig. 3).

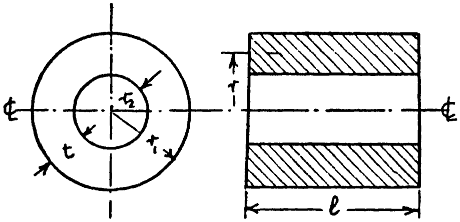

FIG. 3. Dimensions of solenoidal coils.

The value of K_n for solenoidal coils of appreciable winding depth can be calculated from the equation

$$K_n = \frac{1}{1+0{\cdot}9\left(\frac{r}{l}\right)+0{\cdot}32\left(\frac{t}{r}\right)+0{\cdot}84\left(\frac{t}{l}\right)} \quad . \quad . \quad (33)$$

Note that r is now the mean radius r_0 of the coil winding. Examples of the use of this formula will be found in Chapter XIII.

(*c*) Toroids (winding depth small)

Fig. 4 (*a*) shows a cross-section through a coil which is in the form of a toroid (i.e., a circular ring of circular cross-section).

When the ratio $\left(\frac{a}{r_0}\right)$ is very small, the inductance is practically the same as that of a long solenoid for which $r=a$ and $l=2\pi r_0$,

$$\text{i.e., } L=2\pi N^2 r_0 \left(\frac{a}{r_0}\right)^2 . \, 10^{-9} \text{ henries.}$$

As $\left(\frac{a}{r_0}\right)$ is increased, the distribution of field intensity over the cross-section of each turn is no longer uniform, and the mean radius r_0 in the above equation must be replaced by an " effective radius ", given by

$$r_{\text{eff}}=\frac{r_0}{2}\left[1+\sqrt{1-\left(\frac{a}{r_0}\right)^2}\right]$$

The complete expression for the inductance is then

$$L=\frac{\pi N^2 a^2}{r_0}\left[1+\sqrt{1-\left(\frac{a}{r_0}\right)^2}\right]^{-1} . \, 10^{-9} \text{ henries} \qquad (34)$$

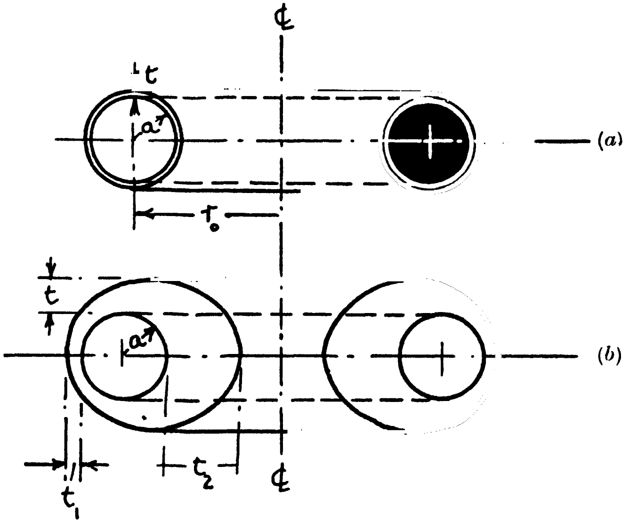

Fig. 4. Dimensions of toroidal coils.

Similar expressions can be derived for the inductance of " toroids " with cross-sections that are not circular, but it is usually found that the effective radius does not differ very much from the mean radius r_0. The inductance can thus be calculated fairly accurately from a general equation of the form

$$\begin{aligned} L &= \frac{4\pi N^2 A}{2\pi r_0} \,.\, 10^{-9} \\ &= \frac{2N^2 A}{r_0} \,.\, 10^{-9} \text{ henries} \qquad . \quad . \quad . \quad . \qquad (35) \end{aligned}$$

where A is the cross-sectional area of one turn.

(*d*) TOROIDS (with deep windings)

As the winding depth t is increased, the problem of calculating the inductance of a toroidal coil becomes increasingly difficult, owing to the changing shape and size of the turns, as the winding process progresses. Fig. 4 (*b*) shows a typical cross-section of such a coil, indicating the eccentricity and distorted shape of the outer turns of the winding. There is no simple formula connecting the inductance and the number of turns, so that the design of a coil of this type becomes largely one of trial and error. Some guidance can be obtained from (34) by taking a as the approximate radius of a turn in the middle of the winding and r_0 as the distance of the centre of this turn from the axis of the coil.

10. Eddy-current losses

(*a*) SKIN EFFECT

Consider a long straight wire, which carries an alternating current of constant amplitude and which is not subjected to any externally-imposed magnetic field. The current will set up an alternating magnetic field, represented by lines of force in the form of circles concentric with the axis of the wire. This field will induce eddy-currents in the wire itself. The direction of these is indicated diagrammatically in Fig. 5 (*a*). One result is that power will be dissipated by the eddy-currents, so that the resistance of the wire to the alternating current will be greater than it would have been to d.c. At the same time, the current distribution in the wire will be changed, because the eddy-currents will tend to oppose the main current in the centre of the wire and augment it near the surface, thus causing the current density to increase towards the surface of the wire. This effect becomes more and more marked as the frequency is raised until, at relatively high frequencies, the current flows almost entirely in a thin outer " skin ", while the current density at the centre of the wire is reduced practically to zero. Under these conditions, since the central portion of the wire does not contribute appreciably to the total current flow, it will make very little difference if it is removed altogether ; so that, at high frequencies, a solid conductor of a given diameter will have roughly the same resistance as a hollow tube of the same material and the same outside diameter. The resistance, in fact, then depends on surface area rather than cross-sectional area, and becomes inversely proportional to the diameter of the wire instead of to the square of the diameter. The absence of current in the central portion of a wire, in which skin effect is occurring, implies a corresponding absence of magnetic field. The eddy-currents, induced in a conductor by an alternating current flowing in it, thus have a magnetic shielding effect which tends to prevent the existence of a magnetic field in the interior of the conductor.

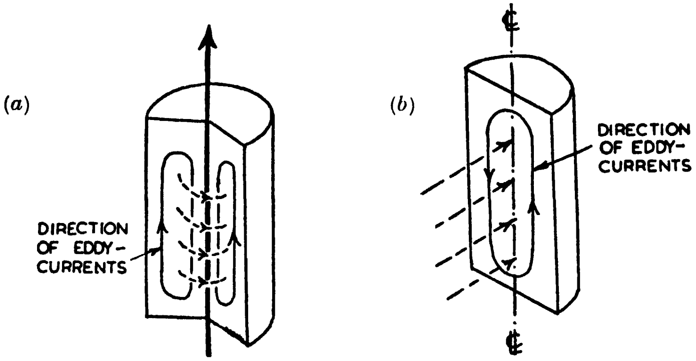

FIG. 5. Diagrams illustrating "skin-effect" and "proximity-effect". (Dotted lines show the direction of the magnetic field in each case.)

It can be shown that, at high frequencies, the current density, at a radial distance x below the surface of a wire of permeability μ and resistivity ϱ ohms-cm., is proportional to $exp.\left(\frac{-x}{\delta},\right)$ where $\delta = \frac{1}{2\pi}\sqrt{\frac{\varrho}{\mu f}}\,.\,10^{4.5}$. The quantity δ is called the "depth of penetration", and can be defined as the depth below the surface at which the current density has fallen to a fraction $\frac{1}{\varepsilon}$ (about 37%) of its value at the surface. At low frequencies, when the depth of penetration is large, the a.c. resistance of the wire is practically the same as its d.c. resistance, but as soon as the value of δ becomes less than about half the wire diameter, the resistance begins to rise rapidly. The resistance of a straight circular wire can be expressed as

$$R = R_{dc}\,(1 + F) \qquad \ldots\ldots \qquad (36)$$

$$\text{or } \frac{R}{R_{dc}} = 1 + F$$

where F is a function of the ratio $\left(\frac{d}{\delta}\right)$. Fig. 6 shows (1 + F) plotted against $\left(\frac{d}{\delta}\right)$. It is seen that at low frequencies F is practically zero, and skin effect is negligible, while at high frequencies (1 + F) becomes very nearly equal to $\frac{1}{4}\left(\frac{d}{\delta}\right)$.

The resistance is given then by

$$R_{hf} = \frac{R_{dc}}{4}\left(\frac{d}{\delta}\right)$$

$$= \frac{\varrho}{\pi d^2}\left(\frac{d}{\delta}\right)$$

$$= \frac{2}{d}\sqrt{\mu\varrho f} \times 10^{4.5} \text{ ohms.}$$

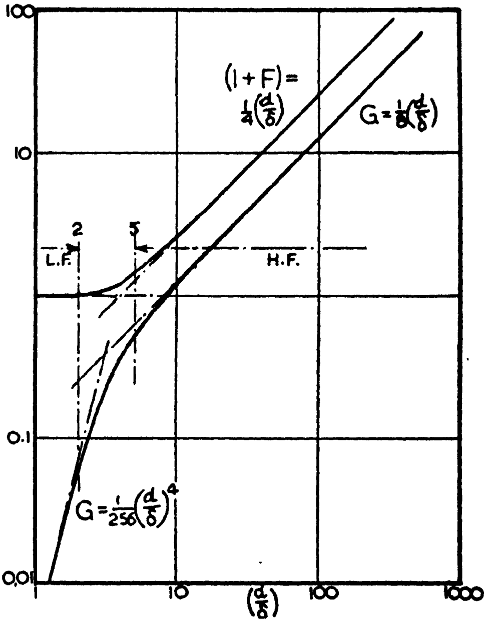

FIG. 6. Curves for calculating skin-effect and proximity-effect losses.

The high-frequency resistance of a wire is thus proportional to $\frac{\sqrt{f}}{d}$ as would be expected from the fact that the current is confined to a skin whose depth is inversely proportional to $\sqrt{f}$ and whose cross-sectional area is therefore proportional to $\frac{d}{\sqrt{f}}$.

Table I shows some typical values of the depth of penetration for various materials.

TABLE I

Material	*Depth of penetration* (*cm.*)
" Nichrome " resistance wire . .	$\frac{50}{\sqrt{f}}$
Brass	$\frac{12 \cdot 7}{\sqrt{f}}$
Aluminium	$\frac{8 \cdot 3}{\sqrt{f}}$
Copper	$\frac{6 \cdot 6}{\sqrt{f}}$
" Mumetal "	$\frac{0 \cdot 4}{\sqrt{f}}$

As an example, the depth of penetration (measured in cm.) for copper is given approximately by $\frac{6 \cdot 6}{\sqrt{f}}$, where f is the frequency in cycles per second. Thus, for a copper wire of, say, 30 S.W.G., the diameter is ·0124 in., (·0315 cm.) so that $\left(\frac{d}{\delta}\right) = \frac{\cdot 0315\sqrt{f}}{6 \cdot 6}$

$$\text{or } f = 4 \cdot 4 \left(\frac{d}{\delta}\right)^2 \times 10^4$$

For this particular wire, therefore, the critical frequency at which $\left(\frac{d}{\delta}\right) = 2$ and skin effect begins to become appreciable is

$$8 \cdot 8 \times 10^4 \text{ c/s}$$

(*b*) PROXIMITY EFFECT

If the wire is now formed into a single circular loop, the diameter of which is large compared with that of the wire, its a.c. resistance will be practically the same as that of an equivalent straight wire. Suppose that another loop is added to the first, forming a two-turn helical coil of close pitch. Any point in the wire is now cut by a magnetic field which is the vector sum of two components, one due to the current in the loop containing the point, and the other due to the current in the other loop. As a result, in addition to the skin effect, the eddy-current loss resistance of either loop is increased by the presence of a magnetic field component caused by the other. This increase of resistance is called the " proximity effect ". The relative magnitudes of skin effect and proximity effect losses in a coil will depend on the number of turns, on the spacing between the turns, and on the shape of the winding. The field which causes a proximity effect cuts the wire practically perpendicularly, and induces eddy-currents whose direction of flow is indicated diagrammatically in Fig. 5 (*b*). The eddy-current power loss in a circular wire, cut perpendicularly by an alternating magnetic field whose r.m.s. intensity over each cycle is H, is

$$\tfrac{1}{4}\,R_{dc}d^2H^2G \times 10^2 \text{ watts}^* \quad . \quad . \quad . \quad . \quad . \quad (37)$$

where G is a numerical factor whose value depends on the ratio between the wire diameter and the depth of penetration. Fig. 6 shows the function G, plotted against $\left(\frac{d}{\delta}\right)$. It is seen that the curve can be split into two distinct regions, separated by a transition region which corresponds to frequencies for which the depth of penetration is comparable with the wire diameter. In the low-frequency region, where the induced field due to the eddy-currents is negligible compared with the perpendicular field H, the value of G is very nearly equal to $\frac{1}{256}\left(\frac{d}{\delta}\right)^4$, so that the proximity effect power loss is then proportional to f^2. At high frequencies, where the induced field has an important effect, and causes " eddy-current shielding ", G is given approximately by $\frac{1}{8}\left(\frac{d}{\delta}\right)$ and the power loss then becomes proportional to $\sqrt{f}$.

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* The apparent discrepancy of a factor of 2 between (37) and the corresponding equation given by Butterworth (in *Experimental Wireless*, 1926, p. 203) is explained by the fact that the latter uses peak values instead of r.m.s. values.

(c) Eddy-current Losses in Stranded Wire

The eddy-current losses in a conductor, carrying an alternating current, can often be reduced by dividing the conductor into a bundle of insulated strands, thus increasing the effective resistance of the eddy-current paths. Let us consider more carefully the conditions under which the maximum improvement can be obtained. The first important point is that mere subdivision of the conductor into a bundle of insulated strands is not sufficient; in fact, if the strands are all parallel and untwisted, and the thickness of the insulation separating them is negligibly small, the bundle will have practically the same losses as a solid wire of the same outside diameter. It can be seen from Fig. 5 (*a*) and (*b*) that both skin effect and proximity effect will take place in the bundle of strands as a whole, and that there will be a tendency for the current to concentrate in some strands at the expense of the current flow in the others. Thus, to be effective, a stranded conductor must be designed in such a way that no circulating currents can flow from one strand to another through the terminations of the wire; i.e., so that each strand carries an equal share of the total current. This can be done by interweaving and twisting the strands so that any pair of strands are successively transposed relative to the magnetic field which is causing the induced eddy-currents. If there is no externally applied field, the magnetic lines of force, caused by the current in the conductor, will be circles concentric with its axis. Twisting of the whole bundle of strands about its own axis will have no effect on the resulting induced currents, which can be reduced only by interweaving the strands in such a way that the position of each one alternates regularly between the centre and the outside of the bundle. The induced currents, however, produced by a perpendicular external field, will be reduced by a simple axial twist, which regularly transposes the strands relative to the applied field. A more detailed analysis of the problem will be found in Chapter IV. For the moment let us suppose that the strands have in some way been arranged so that circulating currents in the bundle as a whole can be entirely neglected. When this has been done, there remains only the sum of the eddy-current losses which would have taken place in each strand alone and which can be calculated from (36) and (37). Usually the strands will be made sufficiently

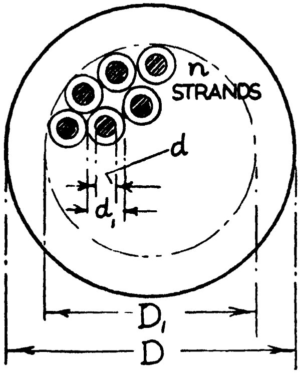

Fig. 7. Cross-section of stranded wire.

thin for the skin effect in each individual strand to be negligible. The total eddy-current loss is caused by the perpendicular field only, and is n times the loss in a single strand.

This loss is given by

$$\tfrac{1}{4}\, n\, R_{dc} d^2 H^2 G \times 10^2 \text{ watts} \quad . \quad . \quad . \quad . \quad (38)$$

where nR_{dc} is the d.c. resistance of one strand.

The total loss for n strands is then

$$\tfrac{1}{4}\, n^2 R_{dc} d^2 H^2 G \,.\, 10^2 \text{ watts} \quad . \quad . \quad . \quad . \quad (39)$$

where R_{dc} is the d.c. resistance of the whole bundle of strands in parallel. This is the minimum possible value of the eddy-current loss, and can be obtained only by choosing the strand diameter so that skin effect in the individual strands is negligible, and also by ensuring that the strands are adequately twisted and interwoven, in order to prevent losses due to circulating currents.

(*d*) TOTAL EDDY-CURRENT LOSS

The above results can be summarized by stating that, for any air-cored coil, there are two distinct ranges of frequency, over each of which simplified eddy-current formulae can be used. These are:

(*i*) *low-frequency* $\left[\left(\frac{d}{\delta}\right)<2\right]$: skin-effect loss is negligible and the proximity effect loss is proportional to f^2.

(*ii*) *high-frequency* $\left[\left(\frac{d}{\delta}\right)>5\right]$: both losses are proportional to $\sqrt{f}$.

The eddy-current losses can thus always be expressed in the form of a series resistance R_e, where

$$R_e \propto f^2 \quad \left[\left(\frac{d}{\delta}\right)<2\right]$$

$$R_e \propto \sqrt{f}\left[\left(\frac{d}{\delta}\right)>5\right]$$

(*e*) EDDY-CURRENT RESISTANCE OF SINGLE-LAYER SOLENOID

Consider first of all the proximity effect between two parallel wires, of diameter d, with their axes separated by a distance c. If the r.m.s. current in each is I, each wire is situated in a field of intensity $\frac{2I}{10c}$ caused by the current in the other wire, so that H in (37) must be replaced by $\frac{2\mathrm{I}}{10c}$. The power loss for both wires then becomes

$$\mathrm{W_e} = \mathrm{R_{dc}} \cdot \frac{d^2 \mathrm{I}^2}{c^2} \cdot \mathrm{G} \text{ watts} \quad . \quad . \quad . \quad . \quad (40)$$

so that the component of the total resistance caused by proximity effect is $\frac{d^2}{c^2} \cdot \mathrm{G} \cdot \mathrm{R_{dc}}$.

The total resistance, including skin effect and proximity effect, is then

$$\mathrm{R} = \mathrm{R_{dc}} \left(1 + \mathrm{F} + \frac{d^2}{c^2} \cdot \mathrm{G}\right) \quad . \quad . \quad . \quad . \quad (41)$$

This equation will hold for all values of $\left(\frac{d}{c}\right)$, provided $\frac{d}{\delta} < 2$. When $\frac{d}{\delta}$ is large, errors will be caused by the fact that the intensity of the field cutting each wire is changed slightly by the change in current distribution in the other wire. In other words, the current in each wire can no longer be assumed to be centred on the geometric axis of the wire. These errors will be most serious for large values of $\frac{d}{\delta}$ and for very close wire spacing. The result obtained for two parallel wires will also apply in the case of a two-turn solenoidal coil, the diameter of which is large compared with that of the wire. As the number of turns is increased, a formula of the same type will apply, but a numerical factor u must be introduced, where u is a function of the number of turns and of the ratio of the length of the coil to its mean radius. The expression for the total resistance of a single-layer solenoid then becomes

$$\mathrm{R} = \mathrm{R_{dc}} \left(1 + \mathrm{F} + u \frac{d^2}{c^2} \mathrm{G}\right) \quad . \quad . \quad . \quad . \quad . \quad (42)$$

As explained above, if the turns are very close together, the problem becomes very complicated because the value of u will then depend also on $\frac{d}{c}$ and on $\frac{d}{\delta}$. Provided the ratio $\frac{d}{c}$ is not greater than about 0·5, Butterworth* gives values of u which are plotted in Fig. 8. For closer spacing than this, Medhurst† has suggested that Butterworth's results are erroneous and he quotes a number of experimental measurements of high-frequency resistance, which can be expressed in the form

$$\begin{aligned} \mathrm{R} &= \mathrm{R_{dc}} (1 + \mathrm{F}) \varphi \\ &= \tfrac{1}{4} \mathrm{R_{dc}} \varphi \left(\frac{d}{\delta}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (43) \end{aligned}$$

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* Butterworth: *Experimental Wireless,* 3, p. 203.

† Medhurst: *Wireless Engineer,* March 1947, p. 35.

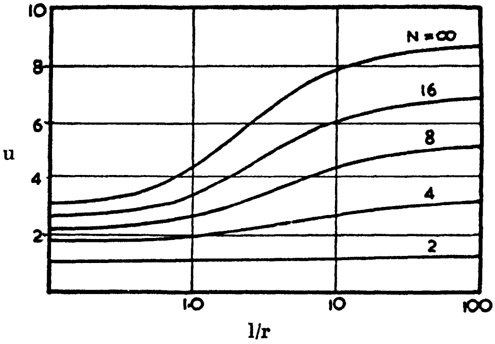

Fig. 8. Curves for calculating eddy current losses in single-layer solenoids (after Butterworth).

where φ is a factor whose value can be obtained from Fig. 9. It is seen that the high-frequency resistance of a given piece of wire will reach a maximum when it is wound into a coil with the turns practically touching, and with its length equal to about 0·7 of its mean diameter. The resistance is then nearly six times as great as that of the same wire when straight, the increase being due, of course, to the proximity effect of the turns of the coil.

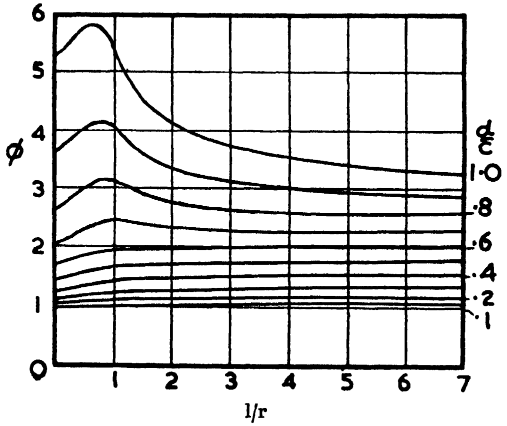

Fig. 9. Curves for calculating eddy-current losses in single-layer solenoids (after Medhurst).

(*f*) Loss Factors

The results already obtained can now be combined in order to obtain relationships between the inductance and the resistance of an air-cored coil. We will deal, first of all, with the case of a multilayer solenoidal coil, operating in its low-frequency region $\left[\text{i.e., } \left(\frac{d}{\delta}\right) < 2\right]$.

The d.c. resistance will be given by

$$4\frac{\varrho l_W N}{\pi d^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (44)$$

where l_W is the mean length of the turns.

In this case, l_W is merely the mean circumference of the winding, so that

$$l_W = 2\pi r$$

$$\text{and } R_{dc} = \frac{8\varrho N r}{d^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (45)$$

Now there is clearly a relationship between the number of turns and the dimensions of the coil and wire. Let us introduce a "space-factor" λ, representing the ratio of the volume occupied by conductor to the total space covered by the winding. The volume of wire is $\frac{1}{2}\pi^2 d^2 N r$, and the winding volume is

$$\begin{aligned} &\pi l (r_1{}^2 - r_2{}^2) \\ &= \pi l (r_1 - r_2)(r_1 + r_2) \\ &= 2\pi l t r \end{aligned}$$

where r_1 and r_2 are respectively the outer and inner radii of the winding, and $r = \frac{1}{2}(r_1 + r_2)$.

$$\text{Then } \lambda = \frac{\pi d^2 N}{4lt} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (46)$$

$$\text{or } d^2 = \frac{4\lambda l t}{\pi N}.$$

Substituting for d^2 in (45),

$$\begin{aligned} R_{dc} &= 8\varrho N r \, . \, \frac{\pi N}{4\lambda l t} \\ &= \frac{2\pi N^2 \varrho r}{\lambda l t} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (47) \end{aligned}$$

Thus, for given winding shape and dimensions, the d.c. resistance is proportional to the square of the number of turns, and it is therefore also proportional to the inductance. But the inductance is given by (28) as

$$L = \frac{4\pi^2 r^2 N^2 K_n}{l} \, . \, 10^{-9}$$

$$\begin{aligned} \text{so that } \frac{R_{dc}}{L} &= \frac{\varrho K_n}{2\pi r t \lambda} \, . \, 10^9 \\ &= K \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (48) \end{aligned}$$

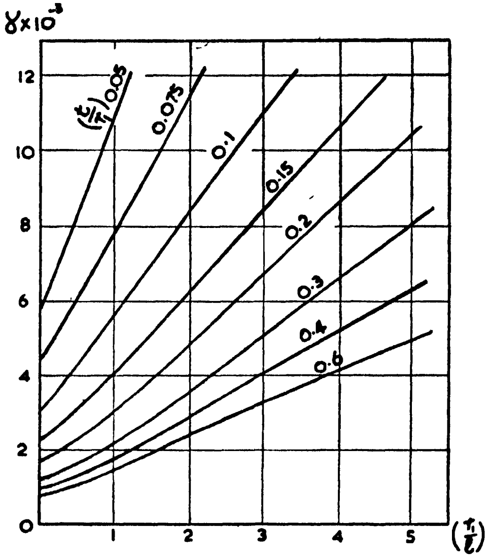

FIG. 10. d.c. resistance of multilayer solenoidal coils.

(Curves for $0.6 < \left(\frac{t}{r_1}\right) > 1.0$ are practically identical with that for $\frac{t}{r_1} = 0.6$.)

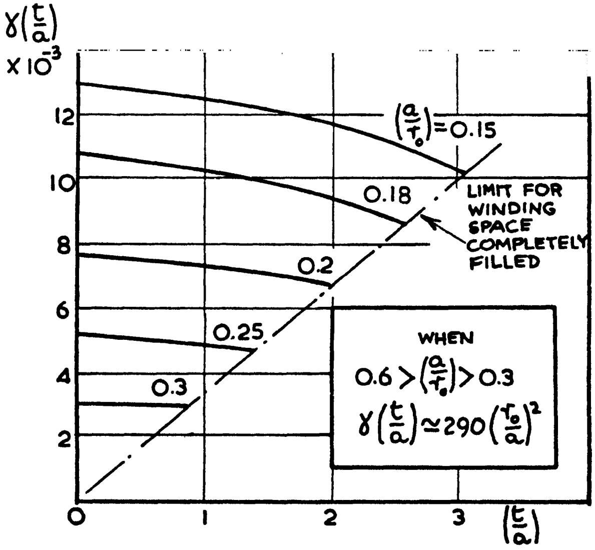

FIG. 11. d.c. resistance of multilayer toroidal coils.

The quantity K is defined as the " d.c. resistance factor " of the coil

$$\text{and } \mathrm{K} = \frac{\varrho \mathrm{K_n}}{2\pi r t \lambda} \cdot 10^9 \qquad (49)$$

For a given winding shape, K is inversely proportional to *rt* ; i.e., to the square of the linear dimensions. It is sometimes more convenient to express K in terms of the outer radius of the coil, by writing

$$\frac{1}{rt} = \frac{1}{r_1^2}\left(\frac{r_1}{t}\right)\left(\frac{r_1}{r}\right)$$

$$\text{then} \quad \mathrm{K} = \frac{\varrho \mathrm{K_n}}{2\pi \lambda r_1^2}\left(\frac{r_1}{t}\right)\left(\frac{r_1}{r}\right) \cdot 10^9 \qquad (50)$$

For a given wire material, ϱ is constant and we can write

$$\mathrm{K} = \gamma \cdot \frac{1}{r_1^2 \lambda} \qquad (51)$$

where γ depends only on the *shape* of the coil. Fig. 10 shows γ, calculated for copper wire, and plotted for solenoidal coils of varying proportions, while Fig. 11 enables the value of γ to be obtained for toroidal air-cored coils, whose dimensions are as shown in Fig. 4 (*b*).

The equivalent series resistance, corresponding to the eddy-current loss, can be obtained by dividing the latter by the square of the current in the winding.

From (37) $\mathrm{R_e}$ is given by

$$\mathrm{R_e} = \tfrac{1}{4}\,\mathrm{R_{dc}} \cdot \frac{d^2 \mathrm{H^2_{mean}}\,\mathrm{G}}{\mathrm{I}^2} \times 10^2 \qquad (52)$$

At low frequencies, G is approximately equal to

$$\frac{1}{256}\left(\frac{d}{\delta}\right)^4 \qquad (53)$$

$$= \frac{\pi^4}{16} \cdot \frac{d^4 f^2 \mu^2}{\varrho^2} \times 10^{-18} \qquad (54)$$

and the eddy-current resistance then becomes

$$\mathrm{R_e} = \frac{\pi^4}{64} \cdot \frac{d^6 f^2 \mu^2 \mathrm{H^2_{mean}}}{\varrho^2 \mathrm{I}^2} \cdot \mathrm{R_{dc}} \times 10^{-16} \qquad (55)$$

but, from (46)

$$d^4 = 16\left(\frac{\lambda l t}{\pi \mathrm{N}}\right)^2$$

and, substituting for d^4 in (55),

$$\mathrm{R_e} = \frac{\pi^2}{4} \cdot \frac{d^2 f^2 \mu^2 \lambda^2 t^2}{\varrho^2}\left(\frac{\mathrm{H_{mean}}\,l}{\mathrm{NI}}\right)^2 \cdot \mathrm{R_{dc}} \times 10^{-16} \qquad (56)$$

The d.c. resistance can be expressed in terms of the inductance by making use of (48), so that

$$R_{dc} = KL$$

$$= \frac{\varrho L K_n}{2\pi r t \lambda} \cdot 10^9 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (57)$$

$$\text{and } \frac{R_e}{L} = \frac{\pi^2}{4} \cdot \frac{d^2 f^2 \mu^2 \lambda^2 t^2}{\varrho^2} \left(\frac{H_{\text{mean}} l}{NI}\right)^2 \cdot \frac{\varrho K_n}{2\pi r t \lambda} \times 10^{-7}$$

$$= \frac{\pi \mu^2}{8\varrho} \left(\frac{t}{r}\right) \left(\frac{H_{\text{mean}} l}{NI}\right)^2 K_n \lambda d^2 f^2 \times 10^{-7} \quad . \quad (58)$$

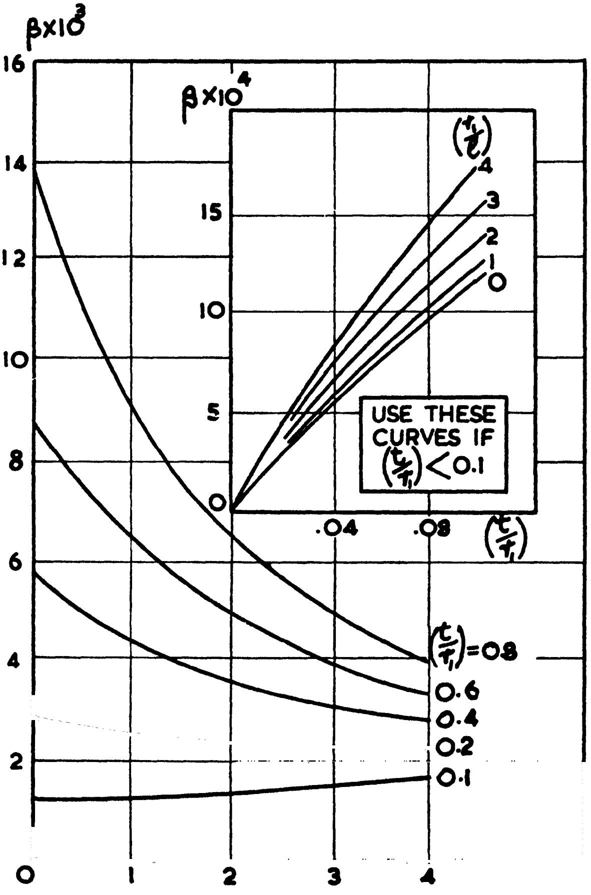

Fig. 12. Eddy-current losses in multilayer solenoidal coils.

Examination of the dimensions of the quantities involved shows that the term $\left(\frac{H_{\text{mean}}l}{NI}\right)$ is a number only, whose value depends only on the shape of the coil. Suppose we simplify (58) by writing

$$\frac{R_e}{L}=\beta\lambda d^2f^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (59)$$

Then, for a given wire material, the value of β depends only on the coil shape. Finally, if the latter is also fixed,

$$\frac{R_e}{L}\propto\lambda d^2f^2$$

$$\text{or } \frac{R_e}{L}=kf^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (60)$$

where k is defined as the "eddy-current factor" of the coil. It will be seen that k has the following properties:

(*a*) it depends on the shape of the coil, but, for geometrically similar coils, is independent of the linear dimensions;

(*b*) it is proportional to λd^2;

(*c*) if the wire material is changed, k is proportional to $\frac{\mu^2}{\varrho}$.

For multilayer solenoidal coils, Butterworth has published expressions from which the quantity $\left(\frac{H_{\text{mean}}l}{NI}\right)$ can be calculated, thus enabling β to be evaluated, for a given wire material, in terms of the shape of the coil. The results have been plotted in Fig. 12. Fig. 13 shows corresponding curves for multilayer toroidal coils. Owing to the difficulty of calculation caused by the eccentricity of the outer layers of the winding as the winding depth increases, the latter values are based on actual measurements of a number of specimen coils of varying proportions. If the coil is wound with stranded wire, composed of n strands, each of diameter d, the space factor will still be the ratio of the volume actually occupied by conductor to the total winding volume,

$$\text{i.e., } \lambda=\frac{Nn\text{ (cross-section of one strand)}}{\text{total cross-section of winding}}$$

and (51) will still hold for the d.c. resistance factor K. Provided the wire is adequately twisted and interwoven, so that losses due to circulating currents are negligible, the eddy-current loss in a multilayer winding, with N turns of the stranded wire, will be the same as it would have been if the winding had consisted of Nn turns of a single strand. In other words, as far as eddy-currents are concerned, the loss depends on the space factor and the strand diameter only, and

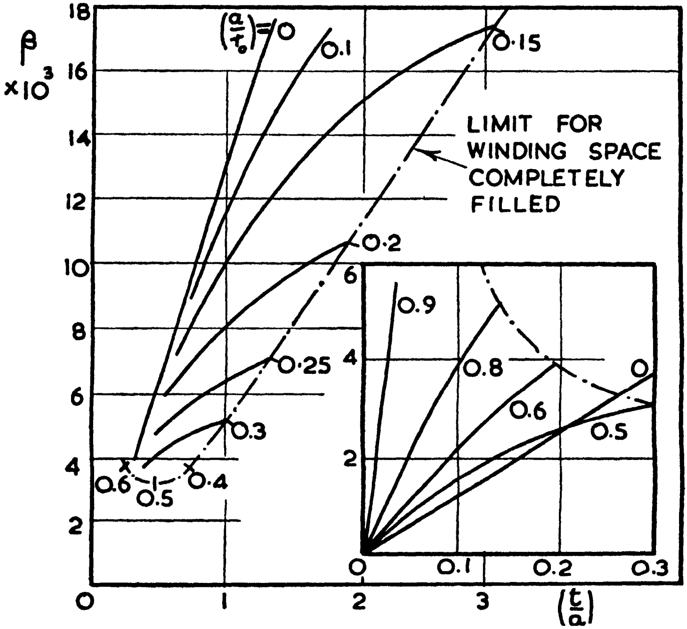

FIG. 13. Eddy current losses in multilayer toroidal coils.

is independent of the number of strands. Thus, using the above definition of λ for the stranded wire, (60) will also apply for multilayer coils wound with stranded wire.

11. Dielectric losses

In addition to the power losses produced by the magnetic field, power is also dissipated in any solid insulating material used in the construction of the coil, resulting from the alternating electric field set up by the potential difference between various parts of the winding. It is an experimental fact that, if this " dielectric " loss is represented by a conductance G, shunted across the coil, G is given by an expression of the form.

$$G = \left(\frac{A}{f} + B\right) \qquad . \quad . \quad . \quad . \quad . \quad . \qquad (61)$$

where A and B are constants for a given size and shape of winding, and for a given insulating material, but where both are independent of the number of turns. If a low-loss material, free from moisture, is used, it is found that the second term is practically negligible, and the first is quite small compared with the other losses in the coil. With the absorption of a relatively small amount of moisture, however, the second term is enormously increased, so that the loss which it represents not only swamps the normal dielectric loss, represented by the first term, but it may also become a large contribution to the total losses in the whole coil. The effect of increased dielectric loss, caused by absorbed moisture, will become more and more marked as the reactance of the coil is increased, i.e., it is most serious for large inductance values at relatively high frequencies. This is a point to be remembered when choosing the $\frac{L}{C}$ ratio of tuned circuits or the design impedance of wave filters. The existence of an excessive loss in the insulation of the wire of which the coil is wound can be checked quite easily by noting whether there is any improvement in Q when the winding is carefully dried out. If it is causing trouble, steps must be taken to seal the coil into a suitable air-tight container, or to replace the wire insulation by a material with less affinity for water. It has been found that, from this point of view, natural silk is very much better than some of the synthetic materials used for wire insulation. It might be as well to point out that the satisfactory sealing of a coil winding against moisture is a procedure which has always offered considerable difficulty, and many methods of dipping and impregnating merely have the effect of delaying the entry of moisture into the insulation. As a result, there is a very real danger that a coil might be produced which would pass its inspection tests satisfactorily but which would then slowly deteriorate, over a period of months, until its Q_0 is perhaps only half the original value. Probably the ideal method is to seal the coil into an airtight can, containing a desiccating agent, and to bring the leads out through proper glands. This has been done on certain equipment for tropical use, although it is obviously too complicated and expensive for general adoption. A good alternative process is to dry the coil winding and then to dip it in a solution of a material such as polystyrene, thus forming a low-loss " skin " over the winding, which is then dipped and re-dipped in melted wax until a thick coating of wax has been formed. The wax then forms the actual seal, while the " skin " of polystyrene prevents the wax from soaking into the winding, where it would cause an increase in both the self-capacitance and the dielectric loss.

12. Screening of coils

Apart from the question of sealing against the entry of moisture, it is often necessary to provide a metal shield, surrounding a coil, in order to reduce coupling between the coil and other electrical circuits. Generally speaking, it is electromagnetic coupling which is the most troublesome, since a metal shield of any form will act as an equi-potential surface and thus considerably reduce capacitance coupling. The magnetic screening effect, on the other hand, depends on the frequency, and on the material and thickness of the screen. It is governed by the shape of the screen and by the "depth of penetration" δ, given by

$$\delta = \frac{1}{2\pi}\sqrt{\frac{\varrho}{\mu f}} \times 10^{4.5} \text{ cm.} \quad . \quad . \quad . \quad . \quad . \quad (62)$$

The depth of penetration, at a frequency f c/s., is about $\frac{6{\cdot}6}{\sqrt{f}}$ cm. for copper and $\frac{8{\cdot}3}{\sqrt{f}}$ cm. for aluminium, so that, at radio frequencies, there is no difficulty in obtaining screens whose thickness is large compared with δ. As the frequency is reduced, the required thickness becomes greater and greater until, at low audio or power frequencies, efficient screens, made of non-magnetic materials, are very bulky and expensive. A glance at (62) suggests that the depth of penetration would be very much reduced if the screen were made of iron or some other ferro-magnetic material with a relatively high permeability. The introduction of iron into the magnetic field of the coil is, of course, accompanied by the usual disadvantages of increased losses and hysteresis distortion, although these are likely to be outweighed by the reduction in the required screen thickness, except in cases where complete freedom from distortion is essential. The operation of an iron screen is only partly due to eddy-current shielding, since, in addition, an opposing field will be produced by the induced magnetic polarity of the screen itself, this effect persisting down to zero frequency, even though the depth of penetration has then become infinite.

To sum up, the material and thickness of the screen should be chosen so that the thickness is at least twice the depth of penetration, over the working frequency range. Copper or aluminium are generally used at radio frequencies, while tinplate is a useful material for screens in the audio frequency range. At sub-audio frequencies, high permeability alloys, such as mumetal, may be used.

The presence of a metallic screen reduces the inductance of a coil and increases its losses, both these effects contributing to a reduction of Q_0. Theoretically, it should be possible to calculate this reduction exactly, but, since this involves obtaining an expression for the intensity of the magnetic field, caused by the current in the coil, at every point in the screen, the labour involved makes the task impracticable except in one or two special cases. We shall here confine ourselves to the statement of some general principles to be observed when choosing the shape and size of a screening-can for a coil. First, it is fairly evident that the losses will be greatest in the parts of the screen which lie at points of greatest magnetic field intensity, suggesting that the most economical screen shape will probably be one in which the intensity is more or less uniform over the whole screen. This means that the screen can be placed quite close to the coil at points where the external field of the coil is small, but should be spaced further away where the field is more concentrated. As a rough guide, a satisfactory shape of screen for a solenoidal coil is a closed cylinder, coaxial with the coil, whose dimensions are such that the clearance between coil and can is nowhere less than the outer diameter of the coil. The effect of screen losses on the Q_0 of the coil will naturally be greatest when the Q_0 is high to start with. The second point is that, for a given ratio of coil volume to volume of screening can, the screen loss will be least for coils whose external field is small; in the case of a toroid, for example, the screen will have very little effect on the Q_0, even when it is practically touching the coil.

A metallic screen increases the apparent self-capacitance of a coil because, in addition to the " internal " self-capacitance caused by the direct capacitance between the turns, there will be an " external " capacitance caused by the screen.

The latter capacitance will increase as the clearance between screen and coil is reduced, and will also depend on whether the screen is connected to one end of the coil or not. In many cases, the eddy-current losses will prevent the screen being brought close enough for the additional capacitance to become serious. This does not apply to toroidal coils, however, owing to their negligible external magnetic field. The spacing of the screen, in this case, may be determined rather by the increase of self-capacitance than by the increased eddy-current losses. As explained elsewhere, the increase in self-capacitance will have the disadvantage of lowering the Q, and also of causing the effective inductance of the coil to vary with frequency. At radio frequencies, large coils in transmitting equipment are sometimes screened by means of a cage of wires rather than by a solid screen. The screening effect of a system of parallel wires depends on their spacing, which must be small compared with the wavelength in use.*

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* Davidson, Looser, Simmonds: *Wireless Engineer*, January 1946, p. 8.

CHAPTER IV

DESIGN OF STRANDED WIRE FOR COILS

13. Definitions

It was shown in Chapter III how the eddy-current losses in coils should be reduced by the use of conductors composed of bundles of insulated strands. To obtain the best results, a mere subdivision of a solid wire* into a number of strands is not sufficient, in fact, if the strands are all parallel, the losses will still be comparable with those of the solid wire. The strands must be twisted and interwoven in such a way that they are continuously transposed relative to the magnetic field cutting the wire. This field can be regarded as made up of two components; a concentric field, caused by the current flowing in the wire at any point, and a perpendicular field, caused by the current flowing in all other turns of the coil. The concentric field will induce currents which will oppose the main current in the central strands of the wire and augment it in the outer strands, producing an effect analogous to the " skin-effect " in a solid wire. This can be reduced by interweaving the strands, so that the position of each one alternates evenly between the centre and the outside of the wire, throughout its length.

In the same way, the circulating currents, induced by the perpendicular field, can be reduced by giving the whole bundle of strands a uniform axial twist. In the following paragraphs, an attempt will be made to derive a simple method of estimating the amount of twisting and interweaving required to obtain adequate suppression of the losses due to circulating currents in the wire.

Fig. 7 shows diagrammatically the cross-section of an idealized conductor consisting of n strands, each of diameter d, each surrounded by insulation which increases its diameter to d_1. The bundle of strands forms a cylinder of diameter D_1, with a further layer of insulation which brings the overall diameter up to D.

14. Axial twist of wire

Suppose that Fig. 7 represents the cross-section of a wire in which the strands are evenly twisted about the axis of the bundle. No current is fed through the wire as a whole, but it is supposed to be cut by a perpendicular magnetic field. Fig. 14 (*a*) shows a longitudinal elevation of the wire, with two typical strands represented by sinusoidal curves. These strands will form a series of loops such as ABCD, which link with the alternating magnetic field applied perpendicularly to the plane of the diagram. Now, in a long solenoidal coil, for example, the intensity of the perpendicular field cutting the conductor can be assumed to vary linearly from zero, for the outer turns, to a maximum value for the inner turns. While this is not strictly true for other types of coil, it is near enough to true for the present purpose, which is merely to obtain a rough estimate of the required amount of twist. It will be assumed therefore that the intensity of the perpendicular field varies uniformly from zero at one end of the wire to a value H at the other end. The field intensity can be assumed to be constant over any loop, and equal to that at the centre-point of the loop. Let this value be H_1.

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* The term " wire " will be used to describe the complete conductor, either solid or stranded, the components of the latter being referred to as " strands ".

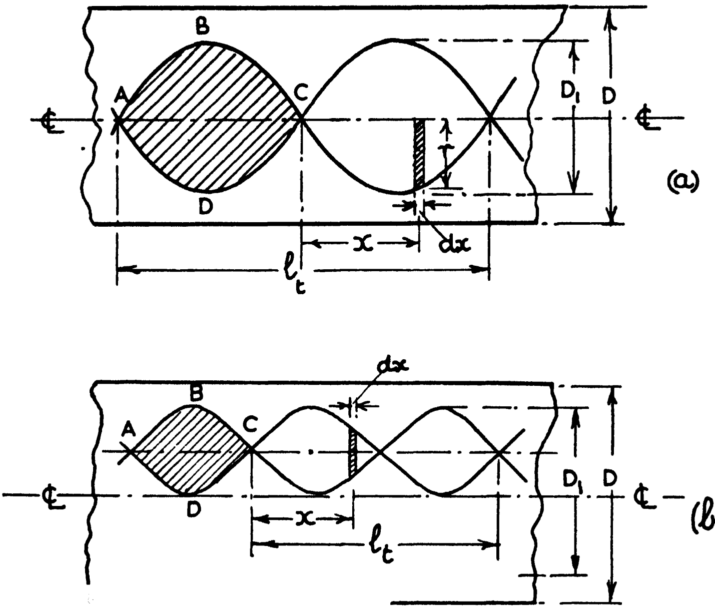

FIG. 14. Diagrams to explain the twisting and interweaving of stranded wire.

$$\text{Then } H_1 = H \cdot \frac{y}{l} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (63)$$

where y is the length of wire between the outer end and the centre of the loop, and l is the total length. The area of any loop is

$$2\int_0^{\frac{1}{2}l_t} r dx = \int_0^{\frac{1}{2}l_t} \frac{D_1}{2} \sin \frac{2\pi x}{l_t}$$

$$= \frac{1}{\pi} D_1 l_t$$

so that the flux linking the loop is

$$\varphi_n = \frac{1}{\pi} D_1 l_t H \frac{y_n}{l},$$

and the flux linking the next loop is

$$\varphi_{n+1} = \frac{1}{\pi} D_1 l_t H \left(\frac{y_n + \frac{l_t}{2}}{l}\right).$$

The difference between these is

$$\varphi_{n+1} - \varphi_n = \frac{1}{2\pi} \cdot \frac{D_1 H l_t^2}{l}$$

This represents the effective flux linking with a complete twist consisting of two adjacent loops. In order to obtain the total flux linking with the two strands, the above expression must be multiplied by the number of twists, i.e., by $\frac{l}{l_t}$, so that the total flux is $\frac{1}{2\pi} D_1 H l_t$ Now suppose that the strands had not been twisted at all. The total flux would then have been $\frac{H l D_1}{2}$. Thus the effect of twisting has been to reduce the flux, and therefore the induced e.m.f. which causes the power loss, in the ratio $\frac{H l D_1}{2}$ to $\frac{1}{2\pi} D_1 H l_t$, i.e., $\frac{\pi l}{l_t}$ to 1 so that the power dissipated is reduced in the ratio

$$\pi^2 \left(\frac{l}{l_t}\right)^2 \text{ to } 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad (64)$$

i.e., $10 \left(\frac{l}{l_t}\right)^2$ to 1 (approx.).

It will be noted that this result is independent of the position of the two strands and that it therefore applies equally well to any pair of strands symmetrically arranged in the bundle. It can therefore be assumed that the axial twisting will reduce the power loss in the whole bundle in the above ratio.

The next step is to obtain an expression for the loss due to circulating currents in a wire with no twisting at all. This would behave approximately like a solid wire, of diameter D_1, but with an increased resistivity ϱ_1, where

$$\varrho_1 = \varrho\frac{D_1^2}{nd^2}.$$

The power loss, due to the perpendicular field, is given by (37) as

$$W_1 = \tfrac{1}{4}R_{dc}D_1^2H^2_{mean}\,G \times 10^2 \text{ watts}$$

$$= \frac{\pi}{16}\varrho_1 D_1^4 H^2_{mean}\,G \times 10^2 \text{ watts} \qquad . \quad . \quad (65)$$

The object of twisting is to reduce this loss until it is negligible compared with the loss produced by eddy-currents in n strands of diameter d. The latter is given by

$$W_2 = \frac{\pi}{16}\varrho d^4 n H^2_{mean}\,G \times 10^2 \text{ watts} \qquad - \quad - \quad (66)$$

and the ratio

$$\frac{W_1}{W_2} = \frac{\varrho_1 D_1^4}{\varrho n d^4}$$

$$= \left(\frac{D_1^2}{d^2}\right) \qquad . \quad . \quad . \quad . \quad . \quad . \quad (67)$$

for example, to reduce $\frac{W_1}{W_2}$ to $\frac{1}{10}$, it is necessary to reduce W_1 by a factor $\frac{10D_1^2}{d_2}$.

Then, from (64)

$$\frac{10D_1^2}{d^2} \overset{(64)}{=} 10\left(\frac{l}{l_t}\right)^2$$

$$\text{giving } \left(\frac{l}{l_t}\right) = \frac{D_1}{d}$$

$$\text{or } l_t = \left(\frac{d}{D_1}\right)l \qquad . \quad . \quad . \quad . \quad . \quad . \quad (68)$$

(68) enables an estimate to be made of the maximum allowable "length of lay" for stranded wire used in coils.

15. Interweaving of strands

Suppose now that Fig. 7 represents the cross-section of a wire in which the strands are evenly transposed throughout the length, in such a way that the position of each strand alternates between the centre and the outside of the wire. Fig. 14 (*b*) shows diagrammatically two typical strands of such a conductor, in which the loops such as ABCD are linked by the magnetic field produced by a current flowing in the whole wire. Since the flux linking any two adjacent loops will be the same, it follows that the total resultant flux linking the two strands may have any value between zero for an even number of loops, and a maximum value, equal to that linking one loop, when the number of loops is uneven. The latter is clearly the worst case, so we will compare the flux linking a single loop with that which would link the two strands if they were straight and parallel.

Suppose that the field intensity varies uniformly from zero at the centre of the wire to a value H at the outside of the wire. Then the average intensity over the element (Fig. 14 (*b*)) will be $\frac{H}{2}$.

The flux linking the element is

$$\frac{H}{2}y\,.\,dx$$

$$=\frac{HD_1 \sin 2\pi x\, dx}{4l_t}$$

Integrating to obtain the flux linking the loop, then flux in loop

$$=\frac{HD_1}{4}\int_0^{\frac{1}{2}l_t} \sin\frac{2\pi x}{l_t}\,dx$$

$$=\frac{HD_1}{4}\cdot\frac{l_t}{2\pi}\left[-\cos\frac{2\pi x}{l_t}\right]_0^{\frac{1}{2}l_t}$$

$$=\frac{HD_1l_t}{4\pi}.$$

If the strands had not been transposed, the area of the loop formed by the two strands would have been $\frac{D_1l}{2}$, and the total flux linkage would therefore have been

$$\frac{H}{2}\cdot\frac{D_1}{2}\cdot l=\frac{HD_1l}{4}.$$

Thus the effect of the transposition is to reduce the flux linkage in the ratio

$$\frac{HD_1l}{4} \text{ to } \frac{HD_1l_t}{4\pi}, \text{ i.e., a ratio of } \frac{\pi l}{l_t} \text{ to } 1.$$

This is the same result as that obtained in the previous paragraph so that, as before, the power loss is reduced in the ratio

$$\pi^2 \left(\frac{l}{l_t}\right)^2 \text{ to } 1 = \pi^2 n^2_t \text{ to } 1 \quad . \quad . \quad . \quad . \quad (69)$$

where n_t is the number of transpositions.

It was pointed out in Chapter III that the losses due to the internal field of a wire are generally more or less swamped by the perpendicular field caused by all the other turns of the coil. Thus it can be seen that a safe rule is to ensure that the " length of lay " of the transposition is not greater than that of the axial twist of the whole wire. It can be shown, in fact, that for coils with more than, say, ten closely wound turns, this form of transposition is not necessary at all, a plain axial twist of the whole wire being all that is required.

16. Mechanical flexibility

Even in cases where transposition of the strands is not necessary from an electrical point of view, wires made up of a large number of enamelled strands may not be sufficiently flexible, and strand breakage may occur when they are wound into coils. In cases of this sort, it is helpful to twist the strands into bundles of, say, nine strands each, and to cover each bundle with a layer of insulating material. These bundles then slide over each other more easily when the complete wire, containing a number of such bundles, is wound into a coil. More complicated methods of interlacing the strands are sometimes used, among them the " Litzendraht " conductor, which is made up of multiples of three strands. Such wires are usually unnecessary and expensive, and have the disadvantage that their space factor is bad ; in any case they would only be justifiable for radio-frequency coils with a small number of turns, where " skin-effect " circulating currents in the wire are more important than circulating currents caused by the perpendicular field. It is often found that such coils can be constructed far more cheaply, and with practically as good a performance, by using solid wire. In other words, where stranded wire is an advantage, simple stranding is sufficient, and more complicated interweaving of strands is necessary only in cases where solid wire could otherwise have been used.

17. Testing of stranded wire

It has been pointed out that a stranded wire must be correctly made up if the full advantages of the stranding are to be obtained. The question immediately arises of finding a method of testing a sample of stranded wire, to determine whether or not it has been adequately twisted, in order to make the losses, caused by circulating currents, negligible. One way in which this can be done is to use the sample to make a "twin-wound" air-cored solenoid; i.e., a coil in which the wire is wound double so that, when completed, the coil consists of two separate, closely coupled windings. The resistance of one winding of the coil is now measured at a frequency high enough to ensure that most of the losses are due to eddy-currents, precautions being taken to see that the wire insulation is kept dry to minimise dielectric losses. First, a measurement is made with the other winding open-circuited, but with all the strands bonded together at each of the two free ends. Then, without changing the measuring conditions in any way, the bonded ends of the second winding are snipped off, and the insulated strands fanned out, so that they cannot come into contact with each other. If the wire has been adequately twisted, there will be no change in the resistance measurement. A drop in measured resistance, however, will indicate that the losses caused by circulating currents, when the ends were bonded, were not negligible. This test deals with the effect of the perpendicular field only. A check of the circulating currents caused by the internal field of the wire can be made by comparing the a.c. resistance of a straight length of the wire with the figure obtained by measuring the average a.c. resistance of one strand, and then dividing this by the number of strands. The two results should agree, within the limits of experimental error, but, if circulating currents are causing trouble, the resistance of the complete wire will be greater than that of the individual strands in parallel.

CHAPTER V

COILS WITH LAMINATED IRON CORES

18. Introduction

This chapter is concerned with the problems involved in the design of coils with cores made up of thin sheets of ferro-magnetic material. These cores may be stamped out of a large sheet, as in the case of the familiar " stampings " used in small chokes and transformers, or the core may be made by winding a continuous metal tape on to a mandrel to form a " toroid " of rectangular cross-section. It is well-known that the reason for using a laminated core instead of a solid mass of iron is that the introduction of insulating layers, running parallel to the direction of the magnetic flux, reduces losses due to eddy-currents induced in the core. Generally speaking, the laminations must be made thinner as the working frequency is raised, and one advantage of the tape cores, mentioned above, is that, by this means, the core can be split into thinner layers than would be convenient with stampings. Another variation of the tape core is a similar construction using iron wire instead of tape.

It must be stressed that the remarks in the following paragraphs refer, unless otherwise stated, to cores with *closed* magnetic circuits, in which, even if an air-gap is used, the length of the gap is so small that its presence simply reduces the flux density without changing its distribution. In other words, " leakage flux " in the air surrounding the core is assumed to be negligible. The more difficult problem of dealing with " open " cores (such as a straight bundle of tapes or wires inserted in a solenoidal coil) will be postponed until Chapter VI.

It is also assumed that the laminations are perfectly uniform. In practice, of course, this can never be so, since the magnetic properties of the material can be controlled only between certain limits, and the rolling and subsequent heat-treatment of the laminations may introduce inequalities and mechanical stresses which affect the results. Deviations from the performance predicted by theory may therefore be expected, but the theoretical analysis is, nevertheless, of great value in establishing general principles, and in indicating the way in which a given design should be modified so as to obtain a desired change in performance.

19. General principles

The magnetic permeability of a material is defined as the ratio $\frac{B}{H}$, where B is the flux density in lines per sq. cm. (gauss) and H is the magnetizing force in oersteds which produced the flux. (1 ampere-turn per cm. is $\frac{4\pi}{10}$ oersteds.)

Consider a uniformly wound, thin, single-layer toroid, completely immersed in a medium whose permeability is μ. The magnetizing force in the core is

$$H=\frac{4\pi NI}{10l} \text{ oersteds}$$

where l is the length of the magnetic path, and the flux density is

$$B=\mu H=\frac{4\pi NI\mu}{10l} \text{ gauss.}$$

The total flux is

$$\Phi=BA=\frac{4\pi NI\mu A}{10l} \text{ lines.}$$

The inductance of a coil is proportional to the number of flux linkages per unit current; i.e., to the product of the number of turns and the total number of flux lines, divided by the current.

Thus

$$L \propto \frac{4\pi NI\mu A}{10l}\cdot\frac{N}{I}$$

$$=\frac{4\pi N^2A\mu}{l}\cdot 10^{-9} \qquad . \quad . \quad . \quad . \quad . \quad . \quad (70)$$

where L is the inductance in henries.

The factor 10^{-9} is necessary to convert from the absolute electromagnetic system of units to the practical system, which is used in defining the henry. Using the above system, the value of μ for free space (and therefore, at a very close approximation, for any non-magnetic material) is unity. The inductance of any coil, when immersed in a medium of infinite extent, will have a value which is exactly proportional to the permeability of the medium. If a finite quantity of iron is brought near to a coil which is otherwise still surrounded by air, the inductance will increase, the ratio of its new value to that without the iron being defined as the " inductance ratio " of that particular arrangement of coil and iron " core ".* This ratio will vary between a value slightly in excess of unity for a relatively small piece of iron, to a value approaching μ for a large mass of iron enclosing the coil. The single-layer toroidal coil, with a close-fitting core, presents an interesting special case. It is found that such a coil, in air, has practically no magnetic field outside the ring. Thus it does not matter what permeability is possessed by the medium outside the coil, and it therefore follows that when the core is in position the coil will have practically the same inductance as it would have had if the iron had extended to infinity. Thus the inductance ratio in this case is equal to the permeability of the core material. The inductance ratio of any other coil and core arrangement will be less than μ.

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* The term " core " will be used to refer to any mass of ferromagnetic material within the field of a coil. This may be actually a core inside the winding, or it may lie outside the winding as well (c.f. " pot-type " cores).

(*a*) COMPARISON OF ELECTRIC AND MAGNETIC CIRCUITS

The formula giving the total flux in the ring-shaped core mentioned above can be written in the form

$$\Phi=\frac{4\pi NI}{10}\cdot\frac{\mu A}{l} \qquad \ldots \ldots \qquad (71)$$

A rough analogy can be drawn between the flux in a closed magnetic circuit and the flow of electric current in a closed conducting loop containing a source of e.m.f. The corresponding formula is

$$I=E\cdot\frac{A\sigma}{l}$$

where A is the cross-section of the wire, l is its length and σ is its conductivity. The quantity $\frac{4\pi NI}{10}$ corresponds to the electromotive force in the electric circuit, and is called the magnetomotive force. The magnetic equivalent of resistance is given by the term $\frac{l}{\mu A}$, and is called the reluctance of the magnetic circuit. The permeability of the magnetic material corresponds to the conductivity of the material of which the wire is made. This analogy is most useful when the magnetic flux is practically confined to an iron core and very little " leakage " takes place into the surrounding air. It should be noted that the permeability of ferromagnetic materials is never more than a few thousand times that of air, and may be much less than this. The magnetic circuit must be compared, therefore, not with the usual well-insulated electric circuit, but with a bare wire immersed in a conducting liquid whose conductivity may become comparable with that of the wire itself. It is quite easy to see that, in this case, there is likely to be a tendency for some of the current to leave the wire and to take " short-cuts " through the surrounding liquid. This will apply to loops of any shape and with any method of applying the e.m.f., except in one particular case: that of a circular loop in which the e.m.f. must be imagined to be uniformly distributed around the loop. From the symmetry of this system it follows that there will be no tendency for the current to leave the wire, whatever the conductivity of the surrounding medium. This will be recognized as the electrical equivalent of the uniformly wound toroidal magnetic circuit which, as already stated, has no external flux, irrespective of the relative permeabilities of the core and the surrounding air. Conversely, for any other shape of core, or even for a toroid, if the winding is not evenly distributed around it there will be a leakage flux outside the core, this leakage becoming more and more important as the core permeability is reduced.

(*b*) EFFECT OF AIR-GAP

Suppose now that the uniform toroidal core has a narrow transverse air-gap, whose length is very small compared with the mean circumference of the ring. The gap will reduce the flux in the same way as a corresponding gap in the electric circuit would increase its resistance and reduce the current flowing in it. If the length of the gap is g, and the total length of the magnetic circuit is $l+g$, the flux is given by

$$\Phi=\frac{4\pi \mathrm{NI}}{10}\div\left(\frac{l}{\mathrm{A}\mu}+\frac{g}{\mathrm{A}}\right) \quad . \quad . \quad . \quad . \quad . \quad (72)$$

This assumes that the flux lines remain practically parallel as they cross the gap, an assumption which is justified only if the gap is relatively short, and the permeability of the core is not too low.

If these conditions are not met, the flux lines will spread out in the gap, producing the so-called " fringing effect " and reducing the effective length of the gap. (72) can be written as

$$\Phi=\frac{4\pi \mathrm{NI}}{10}\cdot\frac{\mathrm{A}\mu}{l+\mu g} \quad . \quad . \quad . \quad . \quad . \quad . \quad (73)$$

$$=\frac{4\pi \mathrm{NI}}{10}\cdot\frac{\mathrm{A}\mu_{\mathrm{g}}}{l}$$

$$\text{where } \mu_{\mathrm{g}}=\frac{\mu}{1+\mu\left(\frac{g}{l}\right)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (74)$$

Thus the gapped core behaves as if it were solid, but with a reduced permeability μ_{g}. The quantity μ_{g} is defined as the " apparent permeability " of the gapped core, and is seen to be a function of μ and of the air-gap ratio $\left(\frac{g}{l}\right)$. The above theory will still apply quite well to closed magnetic circuits of any shape, with any arrangement of the winding on the core, provided the permeability is high and the air-gap ratio is small. Care must be taken to avoid confusion between the *apparent permeability* of a gapped core and the *inductance ratio* (sometimes called the "effective permeability")*, of a coil and core combination.

To illustrate this important point let us consider, as an example, a solid toroidal core made of material with a permeability μ. The mean radius of the ring is r, and its cross-sectional area is A.

First, suppose the core is wound with N turns of wire forming one complete layer. Then, if the radial thickness of the core is small compared with its mean radius, the inductance of this coil will be

$$L = \frac{2N^2A\mu}{r} \cdot 10^{-9} \text{ henries} \quad . \quad . \quad . \quad . \quad (75)$$

If now a narrow radial gap, of length g, is introduced, the inductance will be reduced to

$$L_g = \frac{2N^2A\mu_g}{r} \cdot 10^{-9} \text{ henries} \quad . \quad . \quad . \quad . \quad (76)$$

$$\text{where } \mu_g = \frac{\mu}{1 + \frac{\mu g}{2\pi r}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (77)$$

If the core were removed altogether, the inductance would be

$$L_{a_2} = \frac{2N^2A}{r} \cdot 10^{-9} \text{ henries} \quad . \quad . \quad . \quad . \quad (78)$$

Thus, in this particular case, the inductance ratio of the coil with the gapped core is $\frac{L_g}{L_a} = \mu_g$. Is it therefore exactly equal to the apparent permeability of the gapped core itself. Now suppose that, instead of a complete layer, the winding occupies only a short segment of the core; so short, in fact, that it can be regarded as a straight solenoid of length l where l is small compared with the mean circumference of the core. The inductance with the gapped core will still be given roughly by (76), provided the gap is short and the apparent permeability is not too low. The inductance of the coil without the core, however, will now be L_{a_1}, where

$$L_{a_1} = \frac{4\pi N^2AK_n}{l} \cdot 10^{-9} \text{ henries} \quad . \quad . \quad . \quad (79)$$

---

* The term "effective permeability" is an unfortunate one, and should be avoided in favour of "inductance ratio".

(K_n is Nagaoka's constant—see § 9 p. 26.)

The inductance ratio will be

$$\frac{L_g}{L_{a_1}}=\frac{l}{2\pi r K_n} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (80)$$

and since, by hypothesis, l is smaller than $2\pi r$, it follows that, for any reasonable value of K_n, the inductance ratio will be considerably smaller than the apparent permeability of the core.

(*c*) THE "DEMAGNETIZING EFFECT" OF A GAP

Another way of visualizing the effect of an air-gap in a magnetic core is to regard the reduction of flux as being caused by the induced magnetic polarity in the core on either side of the gap. The reluctance of the core is presumed to be unchanged by the gap, but the magnetizing force is now replaced by an "apparent magnetizing force", made up of the vector sum of the true magnetizing force H and a "demagnetizing force" H′, caused by the induced polarity of the core. In other words, the gap acts roughly as though a very short solenoidal winding had been placed on the core and connected in opposition to the main winding. This argument is sometimes carried a step further, and the effect of a gap is stated in terms of the number of "demagnetizing ampere-turns" which would produce the same result. The corresponding electric circuit consists of a loop with a short gap in it. A reduced electric current will flow round the loop and across the gap, setting up a potential difference across the latter. The current would then be unchanged if the gap were closed again and replaced by a battery whose e.m.f. was exactly equal to the potential drop across the gap and whose polarity was such that it opposed the current flow. (72) can be rearranged in the form

$$\Phi=\frac{4\pi NI}{10}\left(\frac{l}{l+\mu g}\right)\left(\frac{A\mu}{l}\right) \quad . \quad . \quad . \quad . \quad . \quad (81)$$

where the reluctance is still $\left(\frac{l}{\mu A}\right)$, and the apparent magnetomotive force has been reduced to $\frac{4\pi NI}{10}\left(\frac{l}{l+\mu g}\right)$.

(*d*) THE HYSTERESIS LOOP

Consider a sample of iron which has been completely demagnetized and is now subjected to a uni-directional magnetizing force H, whose magnitude increases uniformly from zero. A curve of induced flux density B, plotted against H, will be of the form indicated by the dotted line OQ_1 or OQ_2, according to the polarity of H (see Fig. 15). The permeability of the material is given by

$$\mu = \frac{B}{H}.$$

At a value of H corresponding to the point P_1, for example, μ is thus the slope of the straight line joining P_1 to the origin. μ is clearly a function of H. Its value when H is zero is defined as the "initial permeability" μ_0, and it will increase with increasing H until it reaches a maximum, after which μ again decreases as the core saturates. Now let us suppose that, after increasing from zero to the value corresponding to P_1, H is again reduced smoothly to zero, and then reversed and increased in the opposite direction to a point P_2 at which it is numerically equal to its value at P_1. It is found that the final value of B which is attained is also equal and opposite to its value at P_1. The plotted curve of B against H, however, instead of passing through the origin, is of the form $P_1S_1P_2$. In other words, when H has been reduced to zero, B still has a positive value corresponding to the residual magnetism of the sample. In the same way, as H is taken back from its negative value, through zero, to its original value, the curve $P_2S_2P_1$ is traced. The change of flux again lags

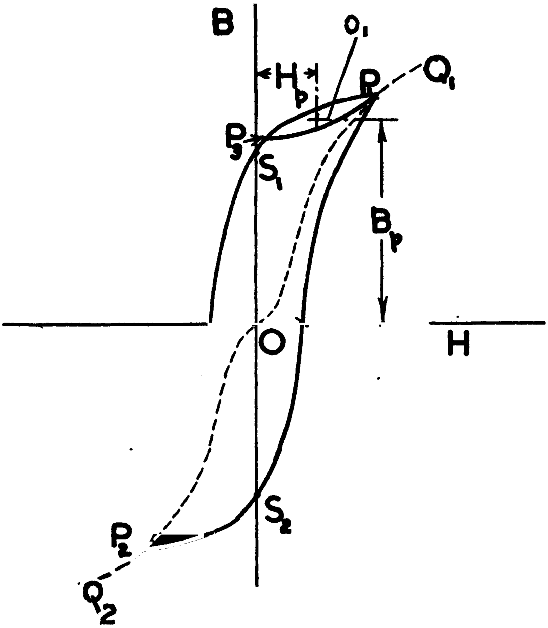


Fig. 15. Diagram representing the hysteresis loop for a ferromagnetic material.

behind the change of magnetizing force, and B has a negative residual value when H passes through zero. This lagging of the flux behind the magnetizing force is called magnetic hysteresis, and the loop $P_1S_1P_2S_2P_1$, obtained by plotting instantaneous values of B and H as H is taken through a complete cycle of alternation, is known as a hysteresis loop. After a few cycles have been completed, the same loop will be traced for each successive cycle, provided the amplitude of variation of H remains the same. Suppose that H varies sinusoidally between maximum values $\pm\hat{H}$, i.e., the instantaneous values of H are given by

$$H = \hat{H} \cos \omega t \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (82)$$

B will vary between corresponding limits $\pm\hat{B}$.

If the magnetization curve of the iron had been quite linear and represented by the straight line P_1OP_2, instead of by the hysteresis loop, the problem would have been quite simple, and the value of B at any instant would have been given by another sine wave in phase with that of H. We should then have had

$$\mu = \frac{B}{H} = \frac{\hat{B} \cos \omega t}{\hat{H} \cos \omega t} = \frac{\hat{B}}{\hat{H}}$$

and μ for the alternating magnetizing force would have been the same as for uni-directional magnetization. In fact, however, owing to the existence of the hysteresis loop, the actual value of B will coincide with that of the ideal sine curve at two points only: the tips of the loop, P_1 and P_2. The waveform of B will be distorted, and will have roughly the form shown in Fig. 16. The curve of B always lies below the ideal curve as H is increasing, and above it as H decreases. The actual B curve can be regarded as made up of two components, one being the ideal undistorted sine wave, and the other a difference waveform, whose ordinates are equal to the difference between the

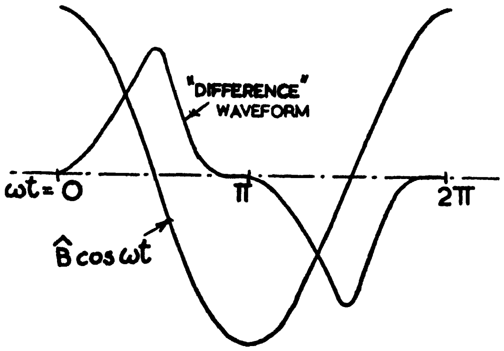


FIG. 16. Waveform distortion caused by hysteresis.

corresponding B values for the real material and an imaginary ideal material with a linear magnetization curve. The question of hysteresis distortion will be dealt with in more detail in Chapter XI. For the present it is sufficient to notice that the difference curve is symmetrical about the axis, and can therefore be analysed into a fundamental wave and a series of *odd*-order harmonic waves. The distorted B waveform can therefore be analysed into

(i) a sinusoidal component, in phase with the applied sinusoidal magnetizing force;
(ii) a sinusoidal component, 90° out of phase with the applied H;
(iii) a series of odd-order harmonics.

The in-phase component of B induces an e.m.f. in the winding, which is 90° out of phase with the current, and therefore represents the inductive part of the coil impedance. The quadrature component of B induces an e.m.f. in phase with the current, corresponding to a resistance in series with the coil inductance, and implying that energy is being dissipated in the core as a result of hysteresis. It can be shown that the energy loss per cycle per unit volume of core is proportional to the area of the plotted hysteresis loop. Consider the simple case of a thin, uniformly wound ring core of cross-sectional area A and mean length l. The magnetizing force produced by an instantaneous current will be

$$\mathrm{H}=\frac{4\pi \mathrm{N}i}{10l}\ \text{oersteds} \quad . \quad . \quad . \quad . \quad . \quad (83)$$

$$i=\frac{10l\mathrm{H}}{4\pi \mathrm{N}}\ \text{amperes} \quad . \quad . \quad . \quad . \quad . \quad (84)$$

The total flux will be BA lines, and the instantaneous induced e.m.f. will be

$$e=\mathrm{NA}\frac{d\mathrm{B}}{dt}\,.\,10^{-8}\ \text{volts} \quad . \quad . \quad . \quad . \quad (85)$$

The power loss is ei watts, so that

$$ei=\frac{\mathrm{A}l\mathrm{H}}{4\pi}\,.\,\frac{d\mathrm{B}}{dt}\,.\,10^{-7}\ \text{watts} \quad . \quad . \quad . \quad . \quad (86)$$

The energy loss per cycle is obtained by integrating over one cycle, i.e., the energy per cycle is

$$\frac{\mathrm{A}l}{4\pi}\int \mathrm{H}d\mathrm{B}\,.\,10^{-7}\ \text{watt-secs. per cycle} \quad . \quad . \quad (87)$$

But $\mathrm{A}l$ is the volume of the core, and the energy loss per cycle per unit volume is thus

$$\frac{1}{4\pi}\int \mathrm{H}d\mathrm{B}\,.\,10^{-7}\ \text{watt-secs. per cycle per cm.}^{3} \qquad (88)$$

In the general case when H and B are not uniform throughout the core, (88) still gives the loss per unit volume, but $\int \mathrm{H}d\mathrm{B}$ is then a variable from point to point in the core, and the total loss for the whole core must be obtained by a further integration.

(*e*) INCREMENTAL PERMEABILITY

The incremental permeability is defined as the ratio of the fundamental alternating component of B to that of the applied H. These two alternating quantities are not exactly in phase with each other, so that the incremental permeability is not a simple ratio, but may be represented by a vector with a modulus $|\mu_\Delta|$, and phase angle φ. An alternative method is to define the incremental permeability as the ratio between the *in-phase* fundamental component of B and the applied H. This amounts to the same thing as a definition of μ_Δ in terms of the ratio between the quadrature component of the induced fundamental voltage and the applied current; i.e., in terms of the "series" inductance of the coil (see § 5).

$$\text{Then } \mathrm{L}=\frac{4\pi\mathrm{N}^2\mathrm{A}\mu_\Delta}{l}.\,10^{-9}\ \text{henries} \quad . \quad . \quad . \quad . \qquad (90)$$

$$\text{or } \mu_\Delta=\frac{\mathrm{L}\,.\,l}{4\pi\mathrm{N}^2\mathrm{A}}\,.\,10^{9} \quad . \quad . \quad . \quad . \quad . \quad . \qquad (90)$$

In some cases it might be convenient to introduce a "shunt" value of incremental permeability μ'_Δ, defined in terms of the shunt inductance L′.

$$\text{Then } \mu'_\Delta=\frac{\mathrm{L}'\,.\,l}{4\pi\mathrm{N}^2\mathrm{A}}\,.\,10^{9} \quad . \quad . \quad . \quad . \quad . \quad . \qquad (91)$$

This latter definition is the same thing as defining μ'_Δ as the ratio between a sinusoidal B and the in-phase fundamental component of the distorted H waveform which produces it. If the incremental permeability is expressed as a vector, its modulus is given by

$$|\mu_\Delta|=\sqrt{\mu_\Delta\mu'_\Delta} \quad . \quad . \quad . \quad . \quad . \quad . \qquad (92)$$

and the phase angle by

$$\tan\varphi=\frac{\mathrm{R}}{\omega\mathrm{L}}=\frac{\omega\mathrm{L}'}{\mathrm{R}'} \quad . \quad . \quad . \quad . \quad . \quad . \qquad (93)$$

where R or R′ is the series or shunt resistance respectively, representing the hysteresis power loss in the core. If I and E are the r.m.s.

values of the fundamental current and induced voltage, the power loss will be

$$P = RI^2 = \frac{E^2}{R'} = \frac{fV}{4\pi}\int H dB \,.\, 10^{-7} \text{ watts} \,. \qquad (94)$$

where V is the volume of the core.

In this book, unless otherwise stated, the "series" value of incremental permeability will be used. μ_Δ is approximately equal to the slope of the straight line joining the tips of the hysteresis loop.

(*f*) EFFECT OF POLARIZING FIELD

Suppose now that, having applied a magnetizing force corresponding to the point P_1, we reduce H until a point such as P_3 is reached (Fig. 15), and then return to the original value of H. A small closed incremental loop will be formed, which will lie wholly above the uni-directional B-H curve. This loop will be traced out during each cycle of a sinusoidal magnetizing force, $H_\Delta = \hat{H}_\Delta \cos \omega t$, applied simultaneously with a steady polarizing magnetizing force H_P. Once again there will be a distorted B waveform, which can be analysed into in-phase and quadrature components, and a series of harmonic waves. In this case, however, the hysteresis loop is not generally symmetrical for positive and negative half-waves, so that *even-order* harmonics may also be produced.

The incremental permeability μ_Δ will still be approximately equal to the slope of the straight line joining the tips of the hysteresis loop.

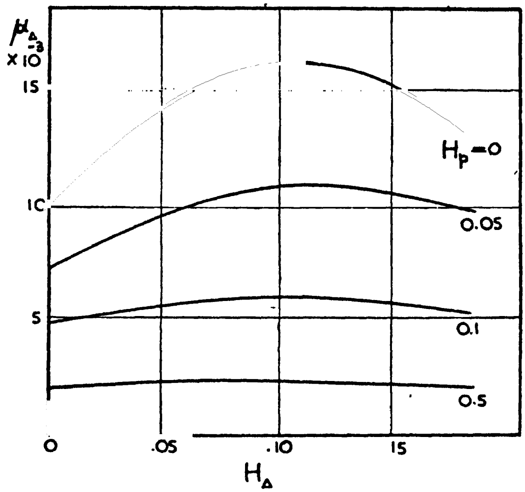


FIG. 17. Typical curves of incremental permeability, plotted against a.c. magnetizing force, for various values of polarizing magnetizing force.

Since P_3 lies on the main hysteresis loop, it follows that the straight line P_1P_3, if extended, must cut the B-axis above the origin, and that its slope must therefore be less than that of OO_1. The slope of OO_1 is μ_p, the permeability of the sample to the polarizing magnetizing force H_p, in the presence of the superposed alternating magnetizing force $\hat{H}_\Delta \cos \omega t$.

$$\text{i.e.,}\quad B_p = \mu_p H_p \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (95)$$

$$B_\Delta = \mu_\Delta \hat{H}_\Delta \cos \omega t \quad . \quad . \quad . \quad . \quad . \quad . \quad (96)$$

and μ_Δ is always less than μ_p.

The mean value of B corresponding to the point O_1 is greater than it would have been if H_p had been applied alone. μ_p is therefore a function of $\hat{H}_\Delta$ and increases as $\hat{H}_\Delta$ increases.

Fig. 17 shows typical curves of μ_Δ, plotted against H_Δ, for various values of H_p. These curves apply to a particular sample of ferromagnetic material, but they will serve to illustrate the general properties which are common to all magnetic alloys.

20. Core eddy-currents at low flux densities

First, we shall assume that there is no d.c. polarization, and that the peak incremental flux density $\hat{B}_\Delta$ (i.e., the maximum amplitude of the alternating flux density), is so small that hysteresis effects can be neglected. Under these conditions, the incremental permeability μ_Δ is independent of $\hat{H}_\Delta$. At very low frequencies, μ_Δ is also independent of frequency, and has a value defined as the " initial permeability ". For the sake of simplicity, in the following paragraphs, the suffix $_\Delta$ will be omitted wherever its omission is not likely to cause confusion. It will be understood that all references here to permeability mean *incremental* permeability (i.e., the ratio of the in-phase fundamental component of the alternating flux density to the alternating magnetizing force which produces it). Instead, the suffix $_0$ will be used to denote an " initial " value ; i.e., the value of a quantity for very low flux densities and very low frequencies. R and L refer to series values and R′ and L′ to shunt values, as defined in Chapter II. Using this notation, it can be shown that, at any frequency,

$$\frac{L}{L_0} = \frac{1}{\theta} \frac{\sinh \theta + \sin \theta}{\cosh \theta + \cos \theta} \quad . \quad . \quad . \quad . \quad . \quad (97)$$

$$\frac{R}{\omega L_0} = \frac{1}{\theta} \frac{\sinh \theta - \sin \theta}{\cosh \theta + \cos \theta} \quad . \quad . \quad . \quad . \quad . \quad (98)$$

$$\frac{L'}{L_0} = \frac{2}{\theta} \cdot \frac{\cosh \theta - \cos \theta}{\sinh \theta + \sin \theta} \quad . \quad . \quad . \quad . \quad . \quad (99)$$

$$\frac{R'}{\omega L_0}=\frac{2}{\theta}\cdot\frac{\cosh\theta-\cos\theta}{\sinh\theta-\sin\theta} \qquad . \quad . \quad . \quad . \quad . \quad (100)$$

$$Q=\frac{\omega L}{R}=\frac{R'}{\omega L'} \qquad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (101)$$

$$=\frac{\sinh\theta+\sin\theta}{\sinh\theta-\sin\theta} \qquad . \quad . \quad . \quad . \quad . \quad . \quad (102)$$

$$\text{where } \theta=2\pi t\sqrt{\frac{\mu_0 f}{\varrho}}\times 10^{-4.5} \; . \qquad . \quad . \quad . \quad . \quad . \quad (103)^1$$

and t is the thickness of the laminations in centimetres.

At first sight it appears as if these unwieldy expressions might be difficult to apply in practice. Fortunately, however, it is found that the complete range of possible values of θ splits into two distinct regions, over each of which it is possible to use much simpler forms. For example, if θ is less than about 1·6, we have, approximately,

$$\left.\begin{aligned} L &= L' = L_0 \\ Q &= \frac{6}{\theta^2} \\ R' &= \omega L' Q = \frac{12\pi}{\theta^2}\,.fL_0 \\ R &= \frac{\omega L}{Q} = \frac{\pi\theta^2}{3}\,.fL_0 \end{aligned}\right\} \qquad . \quad . \quad . \quad . \quad (104)$$

Thus, for $\theta \ngtr 1{\cdot}6$, the inductance of the coil is practically unchanged from its initial value, whilst eddy-currents in the core have introduced a loss which can be represented by a shunt resistance R′. It will be noted that, for a given material and lamination thickness, the value of θ^2 is proportional to f. It thus follows, from (104), that R′ is independent of f. Now, from first principles, the inductance L_0 is given by the equation

$$L_0=\frac{4\pi N^2\mu_0 A}{l}\,.\,10^{-9} \text{ henries},$$

so that, substituting for L_0 in (104) we obtain the following expression for the shunt resistance, in terms of the lamination thickness and the properties of the magnetic material

---

[1] The quantity $\frac{1}{2\pi}\sqrt{\frac{\rho}{\mu_0 f}} \times 10^{4.5}$ is the "depth of penetration" (cf. § 10 (*a*)). θ is thus the ratio of the lamination thickness to the depth of penetration.

$$R' = 12\pi f\left(\frac{\varrho}{4\pi^2 t^2 \mu f}\right)\left(\frac{4\pi N^2 \mu_0 A}{l}\right) \text{ ohms}$$

$$= \frac{12\varrho A N^2}{lt^2} \text{ ohms} \quad . \quad . \quad . \quad . \quad . \quad . \quad (105)$$

Thus R′ is also independent of the core permeability. If the eddy-current power loss is represented by a series resistance R, its value is given by

$$R = \frac{\pi}{3}\,\theta^2 f L_0$$

$$= \frac{4\pi^3 t^2 \mu f}{3\varrho} \,.\, f L_0 \times 10^{-9} \text{ ohms}$$

which can be written as

$$R = F_e f^2 L_0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (106)$$

where F_e is defined as the core eddy-current factor

$$\text{i.e., } F_e = \frac{4\pi^3 t^2 \mu}{3\varrho} \,.\, 10^{-9} = \frac{\pi}{3}\,\theta^2 \,.\, \frac{1}{f} \quad . \quad . \quad . \quad . \quad (107)$$

The value of F_e is a constant for a given material and lamination thickness, and is independent of the size and shape of the core. When dealing with laminated cores, it is usually convenient to express the core eddy-current loss in terms of the shunt resistance R′, since, as shown above, its value is independent of frequency (provided θ is less than 1·6).

(*a*) EDDY-CURRENT SHIELDING

Another set of approximate formulae apply when θ is greater than about 2·5. These are as follows

$$\left.\begin{array}{ll} \dfrac{L}{L_0} = \dfrac{1}{\theta} & \dfrac{R}{\omega L_0} = \dfrac{1}{\theta} \\[2ex] \dfrac{L'}{L_0} = \dfrac{2}{\theta} & \dfrac{R'}{\omega L_0} = \dfrac{2}{\theta} \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad (108)$$

$$Q = \frac{\omega L}{R} = \frac{R'}{\omega L'} =$$

and, substituting for θ,

$$L' = 2L = \frac{1}{\pi t}\sqrt{\frac{\varrho}{\mu_0 f}} \,.\, L_0 \,.\, 10^{4.5}$$

$$= \frac{4AN^2}{tl}\sqrt{\frac{\mu_0 \varrho}{f}} \,.\, 10^{-4.5} \text{ henries} \quad . \quad . \quad (109)$$

$$R' = Q\omega L = \frac{8\pi AN^2}{tl}\sqrt{\mu_0 \varrho f} \,.\, 10^{-4.5} \text{ ohms} \qquad . \quad (110)$$

The first thing to note is that the value of Q has become unity, so that the total Q of the coil, taking the d.c. resistance and other losses into account, will be less than unity. Secondly, it is seen that the inductance, whether expressed as a series or a shunt value, is now inversely proportional to θ. In other words, the inductance is falling as the frequency is raised, and is, in fact, proportional to $\frac{1}{\sqrt{f}}$. This effect, which is equivalent to an apparent decrease in the permeability of the core, is described as " eddy-current shielding " or " magnetic skin effect ". It is analogous to the current skin effect which occurs in an electric conductor, and it is caused by the fact that eddy-currents induce a magnetic flux which tends to oppose the main flux in the interior of each lamination, and augment it near the surface. The apparent decrease in permeability is thus due to the demagnetizing effect of the induced eddy-currents, which " shield " the central portion of the lamination, so that it does not carry its share of the

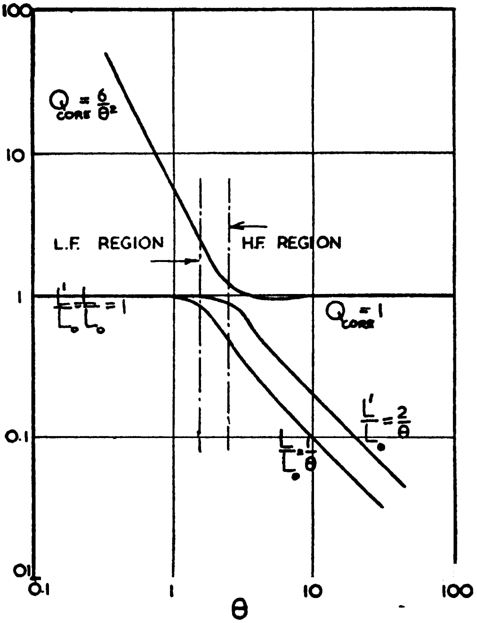


FIG. 18. Relative changes in the inductance and Q of laminated cores, due to eddy-current shielding.

total flux. The shunt resistance is no longer independent of frequency, but, like the shunt inductance, is now proportional to $\frac{1}{\sqrt{f}}$.

(*b*) Critical Frequency

In the transition region where the depth of penetration is comparable with the lamination thickness (i.e., for values of θ between 1·6 and 2·5) it is not possible to use simple approximate formulae. Curves of $\frac{\mathrm{L}}{\mathrm{L}_0}$, $\frac{\mathrm{L}'}{\mathrm{L}_0}$ and Q over this region are plotted in Fig. 18. For a given material and lamination thickness, there is a critical frequency, corresponding to $\theta = 2$, which defines the approximate position of the transition region. The choice of $\theta = 2$ (i.e., depth of penetration equal to half the lamination thickness) is quite arbitrary, but it gives a convenient method of comparing the properties of various laminations. Denoting the critical frequency by f_c, we have

$$2\pi t\sqrt{\frac{\mu_0 f_c}{\varrho}} = 2 \times 10^{4.5}$$

$$\text{or } f_c = \frac{\varrho}{\pi^2 t^2 \mu_0} \cdot 10^9 \qquad \ldots\ldots \qquad (111)$$

Typical values of f_c for some magnetic materials are given in Table II.

TABLE II

CRITICAL FREQUENCIES AT WHICH THE DEPTH OF PENETRATION IS HALF THE LAMINATION THICKNESS

Material	f_c (c/s.)
Mumetal . . .	$\frac{0{\cdot}6}{t^2}$
Radiometal . .	$\frac{2{\cdot}7}{t^2}$
Rhometal . .	$\frac{11}{t^2}$
Stalloy . . .	$\frac{15}{t^2}$

(c) EFFECT OF AIR-GAP

The introduction of a short air-gap has no effect on the equivalent shunt resistance of a coil, since the eddy-current power loss depends on the flux density, and the latter is constant for a given induced e.m.f., irrespective of the length of the air-gap (provided the flux distribution is not upset). Using the suffix $_g$ to denote values with the air-gap, it can be shown that

$$\frac{L'_g}{L_0}=\frac{\frac{2}{\theta}\left(\frac{\cosh\theta-\cos\theta}{\sinh\theta+\sin\theta}\right)}{1+\frac{2\mu_0\beta}{\theta}\left(\frac{\cosh\theta-\cos\theta}{\sinh\theta+\sin\theta}\right)} \quad . \quad . \quad . \quad (112)$$

$$\text{and } Q_g=\frac{R'_g}{\omega L'_g}=\left(\frac{\sinh\theta+\sin\theta}{\sinh\theta-\sin\theta}\right)+\frac{2\mu_0\beta}{\theta}\left(\frac{\cosh\theta-\cos\theta}{\sinh\theta-\sin\theta}\right) \quad . \quad (113)$$

where β is the gap ratio $\frac{g}{l}$.

(112) and (113) define the impedance of the coil in the simplest way. Curves of $\frac{L'_g}{L_0}$ and Q_g are plotted in Fig. 19. The corresponding general expressions for the series inductance and resistance are too complicated to be of much practical use, and will be omitted. It may be as well to recall, in passing, that the impedance of the coil can be completely defined by stating the value of Q, and

the value of either the shunt or series inductance. When the value of Q is small, as it often is in the eddy-current shielding region, it is meaningless to discuss the " inductance " of a coil without specifying whether the series or shunt value is meant. Alternatively, the impedance must be stated as a modulus and angle, the latter approaching 45° in the eddy-current shielding region (θ greater than 2·5). As before, simpler approximate forms can be used, except over the transition range.

(i) *Low Frequencies*

For values of θ less than 1·6, the following equations apply. The usual notation is used, with the suffix $_g$ to denote " gapped " values and the gap ratio denoted by β.

$$\left.\begin{aligned} L_g &= L'_g = L_0\left(\frac{1}{1+\mu_0\beta}\right) \\ Q_g &= \frac{6}{\theta^2}(1+\mu_0\beta) \\ &= Q(1+\mu_0\beta) \\ R'_g &= R' = \frac{12\pi}{\theta^2}\cdot fL_0 \end{aligned}\right\} \quad . \quad . \quad . \quad (114)$$

$$\begin{aligned} R_g &= \frac{2\pi fL_g}{Q_g} \\ &= \frac{2\pi fL_0}{Q}\left(\frac{1}{1+\mu_0\beta}\right)^2 \\ &= \frac{\pi\theta^2}{3}\left(\frac{1}{1+\mu_0\beta}\right)^2 fL_0 \end{aligned} \quad . \quad . \quad . \quad . \quad . \quad (115)$$

The new value of the eddy-current factor will be defined by

$$R_g = F_{e_g} f^2 L_g \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (116)$$

(116) can be written in the form

$$R_g = \frac{\pi\theta^2}{3f}\left(\frac{1}{1+\mu_0\beta}\right)^2 f^2 L_g$$

$$\begin{aligned} \text{whence } F_{e_g} &= \frac{\pi\theta^2}{3f}\left(\frac{1}{1+\mu_0\beta}\right)^2 \\ &= F_e\left(\frac{1}{1+\mu_0\beta}\right) \end{aligned} \quad . \quad . \quad . \quad . \quad . \quad . \quad (117)$$

The eddy-current factor is thus proportional to the apparent permeability of the gapped core.

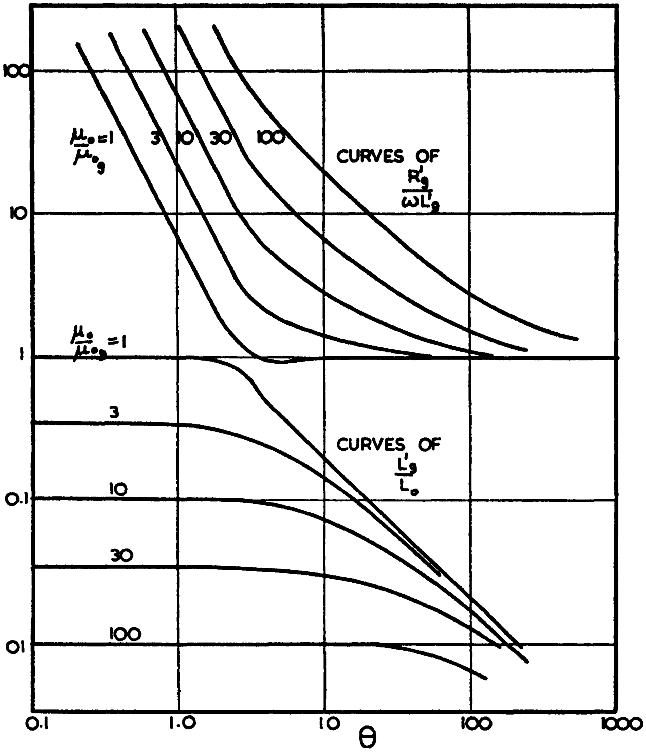


FIG. 19. Extension of Fig. 18 to show the effect of an air-gap in the core.

(ii) *High Frequencies*

In the eddy-current shielding region, the approximate equations are

$$\left.\begin{aligned} \frac{L'_g}{L_0} &= \frac{1}{\frac{\theta}{2}+\mu_0\beta} \\ Q_g &= 1+\frac{2\mu_0\beta}{\theta} \end{aligned}\right\} \quad \ldots\ldots \quad (118)$$

One interesting point about the high-frequency results is that, if θ is large, the introduction of a short gap will have practically no effect at all on the performance of the coil. Although, at low frequencies, an air-gap generally increases the value of Q, this increase becomes progressively smaller as the frequency is raised. As a result, the Q will tend to unity at very high frequencies, even when the core has an air-gap. This would apply, for example, to a laminated core used at radio frequencies or to a solid core (such as a relay core) used at audio frequencies. It must be remembered here, however, that as θ increases and the apparent permeability of the core decreases, there is an ever-increasing tendency for part of the flux to take "short-cuts " through air-paths outside the core rather than through the core itself. When θ is large, and relatively long air-gaps are used, the present theory becomes invalid, and the exact behaviour of the coil is difficult to predict in general terms.

(*d*) INSULATION OF LAMINATIONS

The results of the previous paragraphs are based on the assumption that the laminations are perfectly insulated from each other, so that no eddy-currents cross their surfaces. The surface of the metal is usually painted or treated in some way to produce a non-conducting film between the laminations. If this is not done, the value of θ will be greater than that calculated from (103). This will show up as an increase on the eddy-current factor of the coil, and a reduction in the critical frequency, above which eddy-current shielding takes effect. If the frequency is very high, the problem is further complicated by the fact that, even if perfectly insulated from each other, the laminations will still have a high-frequency leakage path between them, through their mutual capacitance. A consideration of this aspect of the problem will be omitted, partly because it is very unusual for laminated cores to be used at frequencies where this effect becomes apparent, and partly because, in any case, when the flux is concentrated in a very thin " skin " near the surface of the laminations, serious errors are likely to be caused by small variations in the surface conditions.

Example

A numerical example will help to show the type of problem which may arise, and the way in which the results given above can be used to obtain a solution. Suppose that a coil is wound with 1000 turns of wire on a 15-mil. mumetal core. The nominal cross-section of the core is $\frac{3}{8}$ in. × $\frac{3}{8}$ in., and the mean length of the magnetic circuit is 3·05 in., with a 10-mil. air-gap (1 mil. = ·001 in.). What is the approximate Q of the coil at a frequency of 10 kc/sec., if self-capacitance and d.c. resistance can be neglected?

The first step is to calculate the value of θ. Typical values of initial permeability and resistivity for mumetal are :

$$\mu_0 = 7000$$

$$\varrho = 42 \times 10^{-6}$$

$$\text{now } \theta = 2\pi t_\Delta \sqrt{\frac{\mu_0 f}{\ }} \times 10^{-4.5}$$

and $t = \cdot015 \times 2\cdot54 = \cdot0381$ cm.

so that, when $f = 10^4$ c/sec.,

$$\theta = 2\pi \times \cdot0381\sqrt{\frac{7000 \times 10^4}{42000}}$$

$$= 9\cdot8.$$

The gap ratio will be

$$\beta = \frac{g}{l}$$

$$= \frac{\cdot010}{3\cdot05} = 3\cdot28 \times 10^{-3}$$

so that $(1 + \mu_0\beta) = 24\cdot0$.

These values of θ and $(1 + \mu_0\beta)$ can now be used to interpolate in Fig. 19, showing that the value of Q will be approximately 6. This result is independent of the number of turns, and means that, whatever the shunt reactance is, the shunt resistance will be about six times as great. To obtain the actual value of the shunt inductance, the initial inductance L_0 must first be worked out from the equation

$$L_0 = \frac{4\pi N^2 A \mu_0}{l} \,.\, 10^{-9} \text{ henries}$$

$$4\pi \times 10^6 \times 0\cdot92 \times 7000 \times 10^{-9}$$

$$7\cdot65$$

$$= 10\cdot9 \text{ henries.}$$

This is the inductance which the coil would have at a very low frequency, and with no air-gap. The lower portion of Fig. 19 can now be used to determine the ratio $\frac{L'_g}{L_0}$ for the appropriate values of θ and $(1 + \mu_0\beta)$. This gives a ratio of about 0·045, showing that the shunt inductance at 10 kc/sec. and for a 10-mil. gap is :

$$10\cdot9 \times \cdot045 = 0\cdot49 \text{ henries.}$$

The shunt resistance will be six times as great as the reactance,

i.e., $6 \times 2\pi \times 0\cdot49 \times 10^4 = 184{,}000$ ohms.

21. Hysteresis at low flux densities

The hysteresis loop has already been discussed in general terms, and we shall now proceed to consider the shape of the loop in more detail. It was shown by Rayleigh that, provided the value of $\hat{H}$ does not exceed a certain value, the normal magnetization curves of most magnetic materials can be approximated by a curve of the form

$$B = (\mu_0 + aH)\, H \qquad \ldots\ldots \qquad (119)$$
$$\text{or } \mu = \mu_0 + aH$$

where a is a constant for any given material.

Furthermore, not only is the normal magnetization curve of a previously-demagnetized sample represented by the above simple quadratic expression, but the hysteresis loop itself is made up of two parabolic curves. If the magnetizing force alternates between limits $\pm\hat{H}$, the loop equations are

$$\left.\begin{aligned} B &= (\mu_0 + a\hat{H})\, H - \frac{a}{2}(\hat{H}^2 - H^2) \text{ (as H increases)} \\ B &= (\mu_0 + a\hat{H})\, H + \frac{a}{2}(\hat{H}^2 - H^2) \text{ (as H decreases)} \end{aligned}\right\} \quad (120)$$

It has been shown that the energy loss per cycle is proportional to the integral $\int H dB$, taken round the loop, and we can now use the Rayleigh loop equations to obtain an expression for the hysteresis power loss. The area of the Rayleigh loop is given by

$$\int H dB = \frac{4}{3} a\hat{H}^3$$

so that the energy dissipated per cycle per unit volume of core is

$$\frac{a}{3\pi} \cdot \hat{H}^3 \cdot 10^{-7} \text{ watt-secs. per cycle per cu. cm.} \quad . \quad . \quad (121)$$

Generally, when the value of H for a given current in the winding varies from point to point within the core, it is difficult to obtain a relationship between the hysteresis loss and the current. This can be done quite simply, however, in the ideal case where the flux is uniformly distributed over a closed core of mean length l and cross-sectional area A. The magnetizing force H is supposed to be varying sinusoidally between limits $\pm\hat{H}$.

$$\text{Then } \hat{H} = \frac{4\sqrt{2}\pi NI}{l} \cdot 10^{-1} \qquad \ldots\ldots \qquad (122)$$

$$\text{or } \hat{H}^3 = \frac{128\sqrt{2}\pi^3 N^3 I^3}{l^3} \cdot 10^{-3}$$

For an alternating H of frequency f, the average power dissipation is $\frac{a}{3\pi} \cdot \hat{H}^3 fV \cdot 10^{-7}$ watts, where V is the volume of the core.

If this power is represented by a series resistance R, the power must also be equal to RI^2, so that

$$\mathrm{RI}^2 = \frac{a}{3\pi} \,.\, \hat{\mathrm{H}}^3 f\mathrm{V} \,.\, 10^{-7}$$

and therefore, by substituting for $\hat{\mathrm{H}}^3$,

$$\mathrm{R} = \frac{128\sqrt{2}\pi^2}{3} \,.\, \frac{a\mathrm{N}^3\mathrm{IV}f}{l^3} \,.\, 10^{-10} \text{ ohms.} \qquad (123)$$

The next step is to substitute for N^3 from the equation

$$\mathrm{L}_0 = \frac{4\pi\mathrm{N}^2\mathrm{A}\mu_0}{l} \,.\, 10^{-9}$$

$$\text{i.e., } \mathrm{N}^3 = \left(\frac{l}{4\pi\mathrm{A}\mu_0}\right)^{\frac{3}{2}} \,.\, \mathrm{L}_0^{\frac{3}{2}} \,.\, 10^{13.5} \qquad (124)$$

substituting and simplifying, we obtain

$$\frac{\mathrm{R}}{\mathrm{L}_0} = \frac{16\sqrt{2\pi}}{3} \,.\, \frac{af\mathrm{I}\sqrt{\mathrm{L}_0}\mathrm{V}}{(\mu_0\mathrm{A}l)^{\frac{3}{2}}} \,.\, 10^{3.5} \qquad (125)$$

or, since $\mathrm{V} = \mathrm{A}l$,

$$\frac{\mathrm{R}}{\mathrm{L}_0} = \frac{16\sqrt{2\pi}}{3} \,.\, \frac{af\mathrm{I}\sqrt{\mathrm{L}_0}}{\mu_0^{\frac{3}{2}}\sqrt{\mathrm{V}}} \,.\, 10^{3.5} \qquad (126)$$

which can be written as

$$\frac{\mathrm{R}}{\mathrm{L}_0} = \mathrm{F_h}\mathrm{I}\sqrt{\mathrm{L}_0}f \qquad (127)$$

$$\text{where } \mathrm{F_h} = \frac{16\sqrt{2\pi}}{3} \,.\, \frac{a}{\mu_0\sqrt{\mathrm{V}}} \,.\, 10^{3.5}$$

$$= 4{\cdot}2 \,.\, \frac{a}{\mu_0\sqrt{\mathrm{V}}} \,.\, 10^4 \qquad (128)$$

The quantity $\mathrm{F_h}$ is defined as the hysteresis factor of the core. Any coil, with an iron core of any shape, will have a hysteresis factor, defined by (127), but, as stated above, there will not necessarily be a simple relationship between $\mathrm{F_h}$ and the dimensions and properties of the core. In such cases, the hysteresis factor can be measured experimentally, and its value serves as a useful reference for the comparison of the hysteresis effect in various coils.

(*a*) EFFECT OF AIR-GAP ON HYSTERESIS

The above result can be extended to cover the case of a core with an air-gap, provided this merely reduces the flux density without causing an appreciable change of flux distribution in the core. Suppose that the current is kept the same and the value of $\hat{H}$ is reduced to $\hat{H}_g$, where

$$\hat{H}_g = \hat{H} \cdot \frac{\mu_{0_g}}{\mu_0} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (129)$$

The power loss is thus reduced in the ratio $\left(\frac{\mu_{0_g}}{\mu_0}\right)^3$. If the new value of F_h is defined by

$$\frac{R_g}{L_g} = F_{h_g} I \sqrt{L_{0_g}} f \quad . \quad . \quad . \quad . \quad . \quad . \quad (130)$$

$$\text{then } \frac{L_{0_g}}{L_0} = \frac{\mu_{0_g}}{\mu_0} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (131)$$

$$\text{and } \frac{F_{h_g}}{F_h} = \left(\frac{\mu_{0_g}}{\mu_0}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (132)$$

$$\text{Thus } F_{h_g} = \left(\frac{\mu_{0_g}}{\mu_0}\right)^{\frac{3}{2}} \cdot \frac{a}{\mu_0^{\frac{3}{2}}\sqrt{V}} \cdot 4{\cdot}2 \cdot 10^4$$

$$= \frac{\mu_{0_g}^{\frac{3}{2}}}{\sqrt{V}} \cdot \frac{a}{\mu_0^3} \cdot 4{\cdot}2 \cdot 10^4 \quad . \quad . \quad . \quad . \quad . \quad (133)$$

It will be noted that, for a given core material,* the hysteresis factor is proportional to the quantity $\left(\frac{\mu_{0_g}^{\frac{3}{2}}}{\sqrt{V}}\right)$. This is an important result, since it enables the hysteresis factor to be adjusted by choosing any desired core volume and gap ratio.

(*b*) HYSTERESIS COEFFICIENT

It follows also from (133) that when comparing the hysteresis properties of various materials, it is the value of the quantity $\frac{a}{\mu_0^3}$ which should be used as a reference. It is usual to introduce a " hysteresis coefficient " *h*, whose value is defined by

$$h = \frac{8}{3} \cdot \frac{a}{\mu_0^3} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (134)$$

Comparing (133) and (134), the value of *h* is given by

$$F_{h_g} = \mu_{0_g}^{\frac{3}{2}} h \sqrt{\frac{8\pi}{V}} \cdot 10^{3.5} \quad . \quad . \quad . \quad . \quad . \quad (135)$$

---

* Note that V for the gapped core is still A*l*; i.e., V is the volume of iron and does not include the volume of the air-gap. (In any case, the difference between A*l* and A (*l*+*g*) must be quite small, or the present theory would not apply.)

$$\text{or } R_g = \mu_{0_g}^{\frac{3}{2}} L_{0_g}^{\frac{3}{2}} I f h \sqrt{\frac{8\pi}{V}} \,.\, 10^{3.5} \quad . \quad . \quad . \quad (136)$$

This is a convenient form for obtaining the value of ***h***, for the material of a given core, from results of the impedance measurements made on test coils wound on the core. Another form of this expression can be obtained by making the substitution

$$I\sqrt{L_{0_g}} = \sqrt{\frac{4\pi A\mu_{0_g}}{l}} \,.\, NI \,.\, 10^{-4.5} \quad . \quad . \quad . \quad (137)$$

$$\text{then } R_g = \mu^2_{0_g}\left(\frac{4\sqrt{2}\pi NI}{10l}\right) f h L_{0_g}$$

$$= \mu^2_{0_g} \hat{H} f h L_{0_g}$$

$$= \mu_{0_g} h \hat{B}_{0_g} f L_{0_g} \quad . \quad . \quad . \quad . \quad (138)$$

(*c*) INCREASE OF INDUCTANCE WITH FLUX DENSITY

In § 19 (*e*), it was shown that the incremental permeability is given by the slope of the straight line joining the tips of the hysteresis loop. The incremental permeability, in the range of $\hat{H}$ values for which the Rayleigh equations apply, is obtained from (120) by putting $H = \hat{H}$,

$$\text{i.e., } \hat{B} = (\mu_0 + a\hat{H})\,\hat{H}$$

$$\text{so that } \mu = \frac{\hat{B}}{\hat{H}} = (\mu_0 + a\hat{H}) \quad . \quad . \quad . \quad . \quad . \quad . \quad (139)$$

The inductance is therefore given by

$$L = L_0\left(1 + \frac{a\hat{H}}{\mu_0}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad (140)$$

This equation can be obtained in another form by substituting for $\hat{H}$ and N from (122) and (124), respectively,

$$\hat{H} = \frac{4\sqrt{2}NI}{l} \,.\, 10^{-1}$$

$$N = \left(\frac{l}{4\pi A\mu_0}\right)^{\frac{1}{2}} \sqrt{L_0} \,.\, 10^{4.5}$$

$$\hat{H} = \left(\frac{8\pi}{\mu_0 A l}\right)^{\frac{1}{2}} I\sqrt{L_0} \,.\, 10^{3.5}$$

$$\text{but } F_h = \frac{16\sqrt{2}\pi}{3} \,.\, \frac{a}{\mu_0\sqrt{Al}} \,.\, 10^{3.5}$$

$$\text{or } a=\mu_0^{\frac{3}{2}}\sqrt{\mathrm{A}l}\,.\,\frac{3}{16\sqrt{2}\pi}\mathrm{F_h}\,.\,10^{-3.5}$$

$$\text{whence } \frac{a\hat{\mathrm{H}}}{\mu_0}=\frac{3}{8}\mathrm{F_h I}\sqrt{\mathrm{L_0}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (141)$$

$$\text{and } \mathrm{L}=\mathrm{L_0}\left(1+\frac{3}{8}\mathrm{F_h I}\sqrt{\mathrm{L_0}}\right) \quad . \quad . \quad . \quad . \quad (142)$$

(*d*) LIMITATIONS OF RAYLEIGH THEORY

The Rayleigh loop equations apply only over a relatively small range of flux densities, and the results of hysteresis measurements will deviate more and more from the theoretical values as the flux density is increased. This deviation will vary from sample to sample, but it is fairly safe to state that the equations are correct, provided the flux density is not allowed to rise above the value for which the inductance has increased 10% above its initial value ; i.e., provided $\mathrm{F_h I}\sqrt{\mathrm{L_0}}$ does not exceed about $\frac{3}{80}$. When the core has a gap, the limiting value of $\hat{\mathrm{B}}_{\mathrm{g}}$, below which the Rayleigh theory can be applied, is still that for which $\frac{\mu-\mu_0}{\mu_0}=0{\cdot}1$. But, from (114),

$$\mu_{\mathrm{g}}=\frac{\mu}{1+\mu\beta} \text{ and } \mu_{0_{\mathrm{g}}}=\frac{\mu_0}{1+\mu_0\beta} \left(\text{where } \beta=\frac{g}{l}\right)$$

$$\mu_{\mathrm{g}}-\mu_{0_{\mathrm{g}}}=\frac{\mu-\mu_0}{(1+\mu\beta)\,(1+\mu_0\beta)} \quad . \quad . \quad . \quad . \quad . \quad . \quad (143)$$

$$\therefore \frac{\mu_{\mathrm{g}}-\mu_{0_{\mathrm{g}}}}{\mu_{0_{\mathrm{g}}}}=\frac{\mu-\mu_0}{\mu_0}\,.\,\frac{1}{1+\mu\beta} \quad . \quad . \quad . \quad . \quad . \quad . \quad (144)$$

Thus, the limiting flux density for a gapped core is reached, when

$$\frac{\mu_{\mathrm{g}}-\mu_{0_{\mathrm{g}}}}{\mu_{0_{\mathrm{g}}}}=\frac{0{\cdot}1}{1+\mu\beta}$$

$$=\frac{0{\cdot}1}{1+1{\cdot}1\,\mu_0\beta} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (145)$$

It is worth noting that, if the flux density of a *gapped* core is sufficient to raise the inductance of the coil by 10% above its initial value, the core may be working far outside the limits for which the Rayleigh theory applies.

22. Interaction between hysteresis and eddy-current shielding

When the flux density is so great that hysteresis effects cannot be neglected, and, at the same time, the frequency is such that the core is working in the region where eddy-current shielding must be taken into account, the whole problem becomes extremely complicated. The change in flux-density distribution, caused by the shielding effect, will obviously influence the hysteresis loss in the core, whilst the change in incremental permeability as the flux density increases will, in turn, change the induced eddy-currents. Suppose, therefore, we replace (142) and (127) by

$$\frac{L}{L_e} = 1 + \frac{3}{8} F_h I \sqrt{L_0} \,.\, X \qquad . \quad . \quad . \quad . \quad (146)$$

$$\frac{R}{f L_0} = \frac{R_e}{f L_0} + F_h I \sqrt{L_0} \,.\, Y \qquad . \quad . \quad . \quad . \quad (147)$$

where L_e and R_e are the inductance and series resistance, obtained by taking eddy-current effects into account, but neglecting hysteresis. The values of $\frac{L_e}{L_0}$ and $\frac{R_e}{f L_0}$ are then given by the curves of Fig. 19, for various values of θ. X and Y are numerical factors which are both equal to 1 when θ is small, so that (146) and (147) then reduce to (142) and (127).

Much work remains to be done on this subject, but, to show the general tendency, experimental values of X and Y for a particular sample of magnetic material are given in Fig. 20. The same general

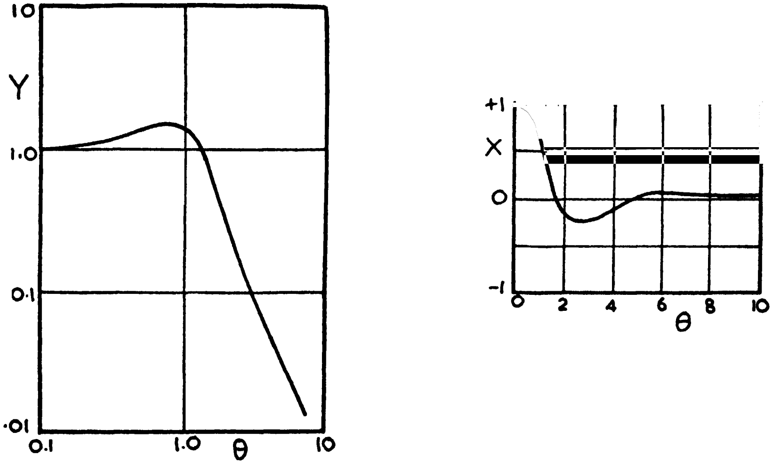
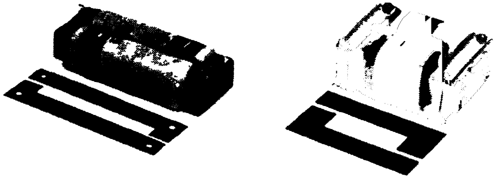
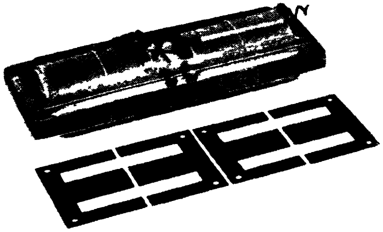


FIG. 20. Typical experimental values of the factors X and Y in equations (146) and (147).



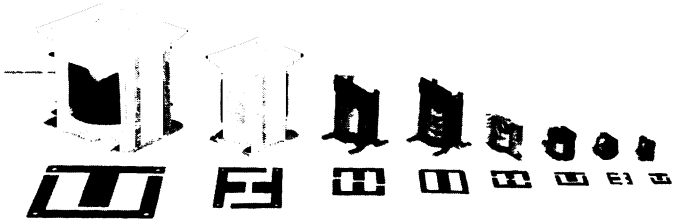
Courtesy Messrs. Smart & Brown, Ltd.



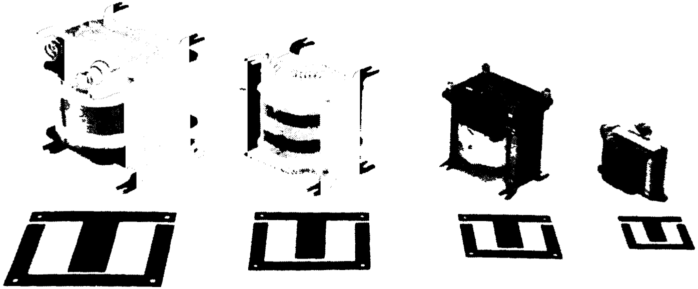
Courtesy Messrs. Micramatic, Ltd.

PLATES I AND II

Typical Control Chokes for Fluorescent Lighting Circuits. (Laminations supplied by Messrs. Magnetic and Electrical Alloys, Ltd.)



Courtesy G.P.O. Engineering Dept.



Courtesy Messrs. Coastal Radio, Ltd.

PLATES III AND IV

Examples of Smoothing and Decoupling Chokes used in Communication Equipment. (Laminations supplied by Messrs. Magnetic and Electrical Alloys, Ltd.)

Plates V and VI

Typical Toroidal Dust cores

Both by courtesy of the Telephone Manufacturing Co. Ltd

Completed Dust cored Inductors, as used in filters, etc behaviour has been found with other materials, although wide variations are often found between different samples of the same material. These variations are probably caused, to a great extent, by the fact that magnetic laminations are never perfectly homogeneous. In spite of care taken during manufacture to ensure that the laminations are uniform in structure, there is always the possibility that the rolling operations will leave a surface " skin ", whose magnetic properties are not exactly the same as those of the inner portion of the lamination. Such inequalities become more and more important as θ increases, and as the flux tends to be concentrated in the surface skin. The curves of Fig. 20 show quite clearly that in the eddy-current shielding region there is no longer a simple relationship between the hysteresis loss and the percentage rise of inductance, even at flux densities for which the Rayleigh theory would have applied at low frequencies.

MATHEMATICAL ANALYSIS

Formulae have been published,* from which the following expressions for X and Y can be derived:

$$X=\frac{L_0}{L_e}\left(1-\frac{4\theta^2}{9\pi}-\frac{7\theta^4}{60}+\frac{2\theta^6}{45\pi}+\ .\ .\ .\right) \qquad . \quad (148)$$

$$Y=\left(1+\frac{\pi\theta^2}{4}-\frac{7\theta^4}{60}-\frac{\pi\theta^6}{40}+\ .\ .\ .\ .\right) \qquad . \quad (149)$$

Comparison with the experimental results of Fig. 20 shows that these expressions give curves which are roughly of the right form, provided θ does not exceed about 1·8. One interesting result, which is predicted by the theory and confirmed by the curve of Fig. 20, is that the value of X passes through zero at $\theta = 1{\cdot}6$ approximately, and then becomes negative. This means that, for any core, there is a certain frequency for which the inductance is practically independent of the flux density. Above this frequency, the inductance will actually *fall* as the flux density increases.

Fig. 21 shows the hysteresis loss per cycle per unit volume, for a certain sample of core material, plotted against peak flux density (for low frequencies). This curve illustrates the relatively small range of flux densities for which the Rayleigh theory applies, and also shows that, at higher flux densities, the hysteresis loss becomes roughly proportional to $\hat{B}^{1\cdot 6}$. This is a general property of magnetic materials, and is known as the Steinmetz Law. The constant of proportionality is called the " Steinmetz coefficient ". Experiment has shown that the hysteresis loss, at any flux density and for any value of θ, is given very approximately by multiplying the ordinates of a curve like

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* Legg: *Bell System Technical Journal,* **15**, p. **39** (**1936**).

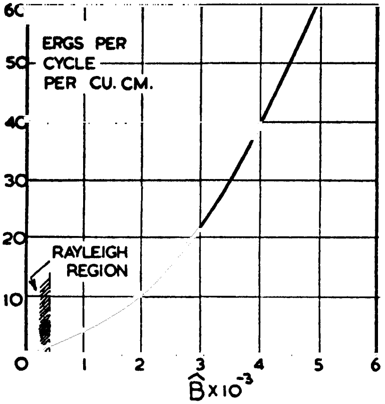


FIG. 21. Typical curve of hysteresis loss, plotted against peak a.c. flux density.

that of Fig. 21 by the appropriate values of the factor Y, obtained from a curve corresponding to that of Fig. 20. In other words, the hysteresis power loss for a given number of ampere-turns in the coil falls off rapidly when θ is greater than 1, this fall in hysteresis loss being much steeper than the drop in inductance caused by eddy-current shielding. This suggests that it might be an advantage to work coils with laminations so thick that θ would be greater than 1 over the working frequency range, but, of course, this reduction in hysteresis loss would only be obtained at the expense of high eddy-current losses and an inductance value which varied with frequency.

23. Effects of d.c. polarization

Iron-cored coils are often used with a d.c. current flowing in the winding. In such cases, we have to take into account the fact that μ_Δ is a function of the polarizing magnetizing force. It will be assumed that eddy-current shielding can be neglected. Fig. 22 shows curves of incremental permeability μ_Δ, plotted against the alternating magnetizing force H_Δ, for various values of polarizing magnetizing force H_P. The curves shown relate to a nickel-iron alloy, but curves of the same general form would apply to any ferro-magnetic material. The presence of the d.c. ampere-turns (or the field produced by a permanent magnet) will reduce the absolute value of μ_Δ and also the percentage variation of μ_Δ with H_Δ.

(a) AIR-GAP THEORY

Coils which are used with d.c. current in their windings are usually provided with gapped magnetic circuits. For a given number of polarizing ampere-turns, the introduction of the gap will reduce the polarizing magnetizing force, thus allowing the incremental permeability to rise. For small gap ratios, this rise will more than outweigh the increase in core reluctance caused by the gap. The inductance of the coil will thus rise. The inductance will continue to rise with increasing gap ratio until the latter reaches an optimum value for which the inductance is a maximum. A further increase in the gap ratio will increase the core reluctance faster than the incremental permeability can rise, and the inductance will start to fall again. Typical curves of incremental permeability, plotted against gap ratio, are shown in Fig. 22. These curves demonstrate quite clearly the existence of the optimum gap ratio, and also show that its value increases as the polarizing magnetizing force is increased. H_p is the total polarizing magnetizing force applied to the gapped core as a whole; i.e., it does not take into account the demagnetizing effect of the gap. Stated another way, H_p is the value of the magnetizing force when the gap is zero.

$$\text{Thus } H_p = \frac{4\pi N I_p}{10 l}$$

The actual value of the magnetizing force at a point in the core would be denoted by H_{p_g} and given by the equation

$$H_{p_g} = H_p \left(1 + \mu_p \cdot \frac{g}{l}\right)^{-1}$$

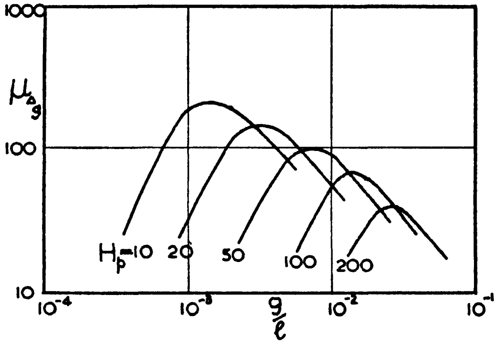


FIG. 22. Typical curves of incremental permeability, plotted against the air-gap ratio, for various values of the polarizing magnetizing force.

(b) GAP RATIO FOR MAXIMUM INDUCTANCE

A problem which often arises is that of designing a coil to have the maximum possible inductance to a small alternating current, in the presence of a comparatively large d.c. component. A typical example is an anode decoupling choke, which must carry the anode current of a valve and also provide the necessary impedance to the alternating signal component. The number of turns must be kept as small as possible, in order to keep the d.c. resistance low; otherwise the coil will have an undesirable d.c. voltage drop across it, whilst the heat produced may also cause difficulties. It is therefore necessary to choose the optimum gap which gives the highest possible

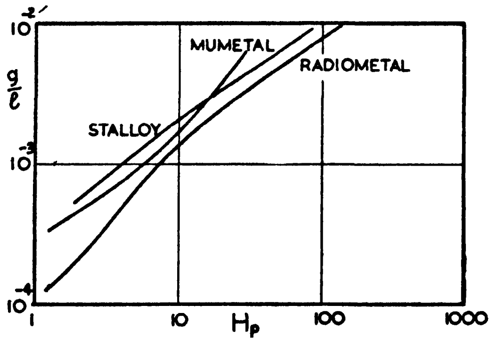


FIG. 23. Curves giving the optimum gap ratio for various values of polarizing magnetizing force. (Like those of Fig. 24, these curves are included to illustrate the method of design; they do not necessarily represent the exact properties of the materials concerned.)

inductance for a given number of turns. For this purpose, the properties of the core material can best be presented in the form of curves such as those shown in Figs. 23 and 24. These show, for three typical materials, the optimum gap ratio, and also the corresponding maximum incremental permeability, for various values of total polarizing magnetizing force H_p. The use of these curves will be illustrated by an example.

Example

A coil has 10,000 turns and is wound on a mumetal core which has a cross-sectional area of 1 sq. cm. and a mean length of 10 cm. What will be the maximum inductance obtainable if the winding has to carry a current of 10 milliamperes d.c., and what should the gap be?

The value of H_p is

$$H_p = \frac{4\pi N I_p}{10 l}$$

$$= \frac{4\pi \, . \, 10^4 \, . \, 10^{-2}}{10} = 12{\cdot}6 \text{ oersteds.}$$

From Fig. 24, the appropriate value of $\hat{\mu}_{\Delta g}$ is found to be about 350. The series inductance of the coil would then be

$$L = \frac{4\pi N^2 A \hat{\mu}_{\Delta g}}{l} \, . \, 10^{-9}$$

$$= \frac{4\pi \times 10^8 \times 350 \times 10^{-9}}{10} = 44 \text{ henries.}$$

The optimum gap ratio is $2{\cdot}0 \times 10^{-3}$, so that the gap length should be adjusted to $2 \times 10 \times 10^{-3}$ cm.

$$= 2{\cdot}0 \times 10^{-2} \text{ cm.}$$

The above curves are based on the assumption that B_Δ is small because the value of μ_{Δ_g} is, strictly speaking, also a function of B_Δ. The error introduced by neglecting B_Δ will not generally be large, and will become less as the gap ratio is increased. In any case, the simplest method of design is to make a preliminary estimate, neglecting B_Δ, and then to make a correction for its effect by trial and error.

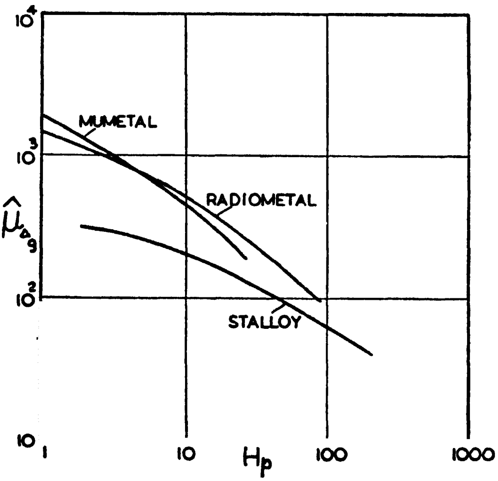


FIG. 24. Curves giving maximum incremental permeability (obtained with the optimum air-gap ratio), for various values of polarizing magnetizing force.

(c) EFFECT OF D.C. POLARIZATION ON CORE LOSSES

As far as eddy-currents are concerned, the effect of d.c. polarization is merely to reduce the incremental permeability, thus reducing the value of F_e. The eddy-current loss, expressed as a shunt resistance, is unaffected by d.c. polarization. To a first approximation, provided the polarizing flux density is not too large, the hysteresis factor also will depend only on the permeability, and will be reduced in accordance with (133). Thus, approximately,

$$F_{e_p} = F_e \left(\frac{\mu_{\Delta p}}{\mu_\Delta}\right)^3 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (150)$$

At high polarization flux densities, the hysteresis effects are complicated by the fact that the shape of the hysteresis loop is also a function of the polarization.

24. Residual loss

At low flux densities and low frequencies, the effective series resistance of a coil, caused by eddy-currents and hysteresis, can be expressed as

$$F_e f^2 L + F_h I\sqrt{L}\,.\,fL.$$

It is found experimentally, however, that the measured resistance apparently includes a term which is proportional to frequency and inductance, but which is independent of the flux density. The corresponding extra loss is called the residual loss and is expressed in the form

$$R_c = F_c fL \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (151)$$

where R_c is the additional series resistance component and F_c is defined as the residual loss factor. The total series resistance, due to core losses, is then

$$\begin{aligned} R &= F_e f^2 L + F_h I\sqrt{L}\,.\,fL + F_c fL \\ &= fL\,(F_e f + F_c + F_h I\sqrt{L}) \quad . \quad . \quad . \quad (152) \end{aligned}$$

The physical significance of the residual loss, which may make a considerable contribution to the total losses in a coil, is not fully understood. The fact that it can be reduced by annealing of the iron and chemical etching of the surface to remove the skin produced by rolling, appears to support the theory that residual loss is connected in some way with energy dissipated in the core as a result of mechanical stresses caused by magneto-striction. If the core has an air-gap, the new value of the residual loss factor is F_{cg}, where

$$\frac{F_{c_g}}{F_c} = \frac{\mu_{\Delta g}}{\mu_\Delta} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (153)$$

The behaviour of residual loss in the eddy-current-shielding region need not be considered, since it is always negligible in comparison with the eddy-current loss.

25. Summary of results

It may be helpful, at this stage, to summarize the more important of the results given in this chapter. For example, it has been shown that an air-gap may be introduced into a laminated core for several different reasons, and that the properties of a coil, wound on a gapped core, depend to a great extent on the gap ratio. The air-gap may be necessary because the coil is to carry a d.c. current in addition to the a.c. component, or it may be used to increase the Q value to a maximum by reducing the core losses at the expense of those in the winding. An air-gap also reduces the hysteresis factor of a coil, so that, since the hysteresis factor is a measure of the harmonic distortion produced by the coil, the use of a large air-gap will reduce harmonic distortion, possibly at the expense of Q. Yet another advantage of the air-gap is its stabilizing effect on all the properties of the coil, so that gapped cores are sometimes necessary merely to stabilize the inductance against variations in the permeability of the core material. This applies particularly to high-permeability magnetic alloys, whose permeability may change considerably with time or as a result of accidental electrical or mechanical shocks.

It has been shown that the eddy-current factor is reduced as the lamination thickness is reduced, while its value is independent of the shape or size of the core. The hysteresis factor, on the other hand, is independent of the lamination thickness, but is inversely proportional to the square root of the core volume. Thus, for a given material, it is possible to change the eddy-current factor by choosing the lamination thickness, and then to adjust the hysteresis factor by changing the core volume. In addition, both factors can be reduced at will by increasing the air-gap ratio. As the latter is increased indefinitely, it is clear that the core loss factors must tend to zero. Unfortunately, however, as soon as the air-gap is made so large that the flux distribution in the core is affected by the presence of the gap, the simple magnetic circuit theory breaks down, and prediction of the performance of a coil with a core, whose *material* properties are known, becomes very difficult. The general problem of design, when the air-gap is large and the permeability of the core material is low, will be referred to again in Chapter VI, in connection with dust-cored coils.

CHAPTER VI

COILS WITH DUST CORES

The term "dust-core" is used to describe a composite material made up of ferro-magnetic particles which have been coated with an insulating "binder" and compressed to form a uniform mass. Such a material as a whole will possess magnetic properties which depend upon, but are quite different from, those of the particles of which it is composed. The magnetic flux in a core of this type passes alternately through magnetic particles and non-magnetic gaps between the particles, so that, even if the material is formed into a closed core with no air-gap, there is still an effective "gap" which is distributed uniformly throughout the core. The "gap-ratio" is of the order of the ratio between the length of the gap between adjacent particles and the mean particle diameter, and therefore reaches a minimum, for a given particle size, when the thickness of the insulating layer between the particles is of molecular dimensions. This leads to the important conclusion that a dust-core behaves in the same way as a gapped laminated core, whose effective gap ratio may be made as large as desired by gapping the core itself, but which cannot be made less than a certain minimum which depends on the particle size and shape. The permeability of a dust-core material is therefore considerably less than that of the particles, and can be expressed as a function of the latter and of the "packing factor"; i.e., the ratio between the volume of metal and the total volume of the core. Fig. 25 shows a typical curve of dust-core permeability, plotted against the packing factor, for a given particle permeability. Even when the packing factor is nearly unity, and a very large proportion of the total volume is occupied by magnetic material, the air-gap effect due to the insulation between the particles is still sufficient to limit the dust-core

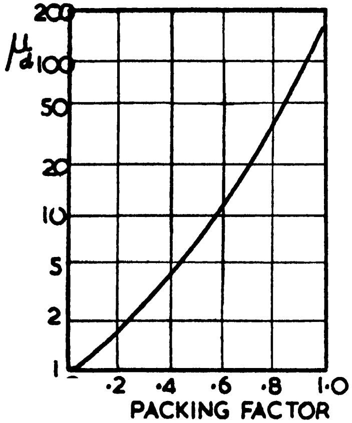


FIG. 25. Variation in the permeability of a typical dust-core material as the packing-factor is changed.

permeability to a maximum of about 150, even when the particles are composed of a high-permeability alloy such as permalloy, and the insulating layers between the particles have been reduced practically to molecular thicknesses. Thus even a completely closed dust-core, such as a toroid, behaves in the same way as a laminated core with extremely thin laminations and with an air-gap which cannot be reduced below a gap ratio of about $\left(\frac{1}{150}\right)$. The dust-core has two advantages over the corresponding laminated core with the same gap ratio; one is that the magnetic material is much more finely divided, so that the effect of very thin laminations can be obtained, and the other is that the "air-gap" is uniformly distributed throughout the core, thus reducing the leakage flux.

26. Eddy-currents in a dust-core material

In § 20, p. 67 it was shown that, at low frequencies, the eddy-current factor of a laminated core is given by the equation

$$F_{e_g} = \frac{4\pi^3 t^2}{3\varrho} \cdot \mu_{0_g} \quad . \quad . \quad . \quad . \quad . \quad (154)$$

For an idealized dust-core material, consisting of spherical particles which are perfectly insulated from each other, the corresponding equation is

$$F_{e_g} = \frac{2\pi^3 t^2}{5\varrho r^{\frac{1}{3}}} \cdot \mu_{0_g} \quad . \quad . \quad . \quad . \quad . \quad (155)$$

where t^2 is now the mean square of the particle diameters and r is the packing factor. In a real dust-core material, these ideal conditions cannot be met, and it is always found that the eddy-current losses are somewhat greater than suggested by (155). Furthermore, as the packing factor is reduced indefinitely, the apparent permeability of the dust-core, with its multiple air-gaps between the particles, must tend to unity. The eddy-current loss must, however, tend to zero, and not to an infinite value as suggested by (155). This condition can be satisfied by introducing an "eddy-current coefficient" e, such that

$$F_{e_g} = e\,(\mu_{0_g} - 1) \quad . \quad . \quad . \quad . \quad . \quad (156)$$

where e is a constant for a given dust-core material, and depends on the size, shape, resistivity and packing factor of the particles of which it is composed.

Dust-cores are generally used at low flux densities, where the permeability of the particles has risen only slightly above its initial value, so that the change in the apparent "gapped" permeability of the material with changes in flux density is negligible. In this chapter, therefore, the suffixes $_0$ and $_g$ will be omitted, and the symbol μ_d used to represent the dust-core permeability, the value of which is assumed to be independent of flux density and frequency. (156) then becomes

$$F_e = e\,(\mu_d - 1) \qquad . \quad . \quad . \quad . \quad . \quad . \quad (157)$$

Provided that the packing factor is not made too small, μ_d will be large compared with 1, and e will be practically independent of r, so that under these conditions F_{c_g} is roughly proportional to the permeability of the dust-core material and to the mean square of the particle diameters.

EDDY-CURRENT SHIELDING

The critical frequency, above which the eddy-current shielding effect begins to occur, is much higher for a dust-core material than for the thinnest practicable laminations. It is generally found, in fact, that the critical frequency is well above the highest working frequency of any given dust-core material. In any case, since the permeability of the material is controlled mainly by the packing factor, it follows that, even when eddy-current shielding begins to reduce the apparent permeability of the particles themselves, this will have very little effect on the permeability of the dust-core as a whole. The loss coefficients are thus practically independent of frequency, and it can also be assumed that no interaction takes place between eddy-currents and the hysteresis properties of the material.

27. Hysteresis

The fact that the permeability of a dust-core is practically independent of frequency and of flux density has led to a different method of expressing the hysteresis loss. This is to introduce a "hysteresis coefficient", whose value is defined by the equation

$$R = \mu_d h \hat{B} f L \qquad . \quad . \quad . \quad . \quad . \quad . \quad (158)$$

Assuming uniform flux density, the relationship between h for the material and the "hysteresis factor" F_h of the whole coil can be obtained as follows.

By definition

$$R = F_h I \sqrt{L}\,.fL \qquad . \quad . \quad . \quad . \quad . \quad . \quad (159)$$

so that, from (158)

$$F_h = \mu_d h \frac{\hat{B}}{I\sqrt{L}} \qquad . \quad . \quad . \quad . \quad . \quad . \quad (160)$$

$$\text{but } \hat{B} = \frac{4\pi\sqrt{2}A\mu_d}{l} \,.\, 10^{-1} \quad . \quad . \quad . \quad . \quad . \quad . \quad (161)$$

$$\text{and } L = \frac{4\pi N^2 A\mu_d}{l} \,.\, 10^{-9} \quad . \quad . \quad . \quad . \quad . \quad . \quad (162)$$

$$\text{so that } I\sqrt{L} = NI\sqrt{\frac{4\pi A\mu_d}{l}} \,.\, 10^{-4.5} \quad . \quad . \quad . \quad . \quad . \quad (163)$$

$$\text{and } \frac{\hat{B}}{I\sqrt{L}} = \sqrt{\frac{8\pi\mu_d}{Al}} \,.\, 10^{3.5} \quad . \quad . \quad . \quad . \quad . \quad . \quad (164)$$

$$\text{so that } F_h = \mu_d^{\frac{3}{2}} h\sqrt{\frac{8\pi}{V}} \,.\, 10^{3.5} \quad . \quad . \quad . \quad . \quad . \quad (165)$$

This result should be compared with (128), which was obtained for laminated cores. Strictly speaking, the volume V in (128) refers to the actual volume of iron, excluding the air-gap, while, in (165), V refers to the *total* volume of dust-core material, including the gaps between the particles. In a first approximation, however, for the gap ratios and packing factors generally used, the value of h should be practically the same for a given magnetic alloy, whether used as laminations or as dust. In practice, it is usually found that in such cases the hysteresis coefficients of dust-core particles tend to be higher than those of the corresponding laminations, and also to increase as the particle size is reduced. This result, which appears to disagree with the above theory, is probably due mainly to variations in flux density from point to point within each particle. The discrepancy in the value of hysteresis coefficients may be quite considerable, and it is not unusual for a relatively low-permeability dust-core material to have a value of this coefficient as much as five or ten times as great as would have been expected from the performance of laminations made of the same alloy.

As the packing factor of a dust-core material is progressively reduced, the value of F_h must tend to zero as μ_d tends to 1. (165), which suggests a finite value of F_h when μ_d is equal to unity, no longer holds good when the packing factor is small, because it can no longer be assumed that the flux density is uniform over each particle of the core. A closer approximation can be obtained by replacing μ_d by $(\mu_d - 1)$ in (165), then

$$F_h = (\mu_d - 1)^{\frac{3}{2}} h\sqrt{\frac{8\pi}{V}} \,.\, 10^{3.5} \quad . \quad . \quad . \quad (166)$$

28. Residual loss

The residual loss factor of a dust-cored coil with uniform flux density is given by

$$F_c = c\,(\mu_a - 1) \qquad . \quad . \quad . \quad . \quad . \quad . \quad (167)$$

where c is a constant for a given material, and it is called the residual loss coefficient.

29. Loss factors of "thick" toroidal cores

When the ratio of inner and outer diameters of a ring core is large, the flux is no longer uniformly distributed over the core cross-section, but its density is greatest near the inner edge of the ring. The relationship between the loss factors and the core loss coefficients can still be obtained, however, by imagining the core to be made up of a series of thin elemental rings, over each of which the flux density can be assumed to be uniform, and then integrating to obtain the result for the whole core. If this is done, it is found that (157) and (167) will still apply, and that (166) is merely modified by multiplying the right-hand side by a numerical factor P, whose value depends on the shape of the core. Values of P have been calculated for ring cores of rectangular cross-section, and are given in Table III.

TABLE III

$x = \frac{r_1}{r_2}$	1·0	1·2	1·4	1·6	1·8	2·0	2·5	3·0	4·0	5·0	7·0	10
P	1·0	1·01	1·02	1·03	1·04	1·06	1·10	1.15	1·26	1·37	1·55	3·52

where r_1 and r_2 are the outer and inner radii of the ring.

Ring cores generally have a cross-section in the form of a rectangle with rounded corners. These are often called "toroids",* although this term should strictly apply only to rings of circular cross-section. For a given ratio of inner and outer diameters, the value of P is greatest for a rectangular cross-section, and its value is slightly reduced by the rounding of the corners. The values of P given in the above table can thus be regarded as maximum values for any given diameter ratio. In practice, the diameter ratio will hardly ever exceed 2, in which case the error produced by taking P as unity would not be greater than about 6%. It can be concluded, therefore, that the equations already obtained for cores with uniform flux densities will also apply with sufficient accuracy to any ring core with the usual ratios of inner to outer diameters.

* In this book, the term "toroid" has also been used to describe ring cores with cross-sections other than circular ones.

30. Loss factors of toroids with "deep" windings

We will next compare the loss factors of coils wound on two similar toroidal cores, one coil having a negligible winding depth and the other having the available winding space almost filled. The inductance of the first coil will be given by

$$L_1 = \frac{2N^2 A_1 \mu_d}{r_0} . 10^{-9} \quad . \quad . \quad . \quad . \quad . \quad (168)$$

where A_1 is practically equal to the cross-sectional area of the core.* For a given number of ampere-turns, the flux in the core of the second coil will be the same as for the first, though, in addition, there will be some flux lines which link with the outer layers of the winding, but which are wholly outside the core. As a result, the second coil will have a higher inductance, and will behave as though it were made up of the first coil, with a small air-cored coil, representing the effect of the leakage flux, connected in series with it.

$$\text{i.e., } L_2 = L_1 + L_l \quad . \quad . \quad . \quad . \quad . \quad . \quad (169)$$

where L_l is the "leakage inductance", the value of which depends on the winding and is independent of the core. Now suppose that the two cores are removed completely and are replaced by wood or some other non-magnetic material. The factor by which the inductance is reduced in each case will be defined as the "inductance ratio" and denoted by *m*.

It can be seen quite easily that L_1 will be reduced to $\frac{L_1}{\mu_d}$ so that the corresponding value of *m* is

$$m_1 = \mu_a \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (170)$$

The leakage inductance will be unchanged, so that

$$m_2 = \frac{L_1 + L_l}{\frac{L_1}{\mu_d} + L_l} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (171)$$

Thus the inductance ratio of the second coil is always less than μ_d. The core power loss, for a given number of ampere-turns, is the same for both coils, so that the effect of the increased inductance of the deep winding is to reduce the core loss factors. The eddy-current and residual loss factors will be reduced in the ratio $\left(\frac{L_1 + L_l}{L_1}\right)$ to 1 and the hysteresis factor in the ratio $\left(\frac{L_1 + L_l}{L_1}\right)$ to 1. This reduction is not likely to be very great for the values of μ_d and the coil proportions which are generally used ; i.e., the core loss factors are practically the same for any uniform winding on a given core.

---

* Strictly speaking, there is a small correction factor, to allow for the fact that the flux density is not quite uniform.

The removal of the cores in the above case will multiply both the winding loss factors by the appropriate value of μ_d. This is an important point, because it shows that the winding loss factors of a shallow winding will be reduced μ_d times when the core is inserted, but that the improvement due to the core may be considerably less for a deep winding. In fact, for a coil with the winding space filled, the improvement may be only about half of what might be expected.

An example will make this clear. Suppose we have a toroidal core with $r_0 = 2$ cm. and $a = 0{\cdot}4$ cm. If the core is non-magnetic, the value of K, for a fully-wound coil, with $\lambda = 0{\cdot}4$, is about

$$\frac{3500}{4 \times 0{\cdot}4} \doteq 2200 \text{ (from Fig. 11).}$$

When the core is replaced by a dust-core with $\mu_d = 120$, the new value of K will not be

$$K_d = \frac{2200}{120} = 19{\cdot}3$$

but will be more like twice this value.

The winding eddy-current factor of the same coil will be

$$K = \beta \lambda d^2$$

where β is about $1{\cdot}1 \times 10^{-2}$ for a full winding.

Thus, if the strands are 46 S.W.G. ($d = {\cdot}0061$ cm.),

$$\begin{aligned} k &= 1{\cdot}1 \times 0{\cdot}4 \times 37 \times 10^{-8} \\ &= 1{\cdot}62 \times 10^{-7} \end{aligned}$$

and k_d will be, very roughly,

$$\frac{1{\cdot}62}{60} \times 10^{-7} = 2{\cdot}7 \times 10^{-9}.$$

The core eddy-current factor is

$$F_e = \mu e$$

and is of the order of 2×10^{-6}, showing that, for high-permeability dust-cores, k_d can be made quite negligible. If μ_d had been 12, k_d would have been about $2{\cdot}7 \times 10^{-8}$, and F_e about 2×10^{-7}.

Even now, the winding eddy-current loss is only just beginning to have an effect on the performance of the coil, so that by suitable choice of the strand thickness it should always be possible to make the winding eddy-current loss of a toroidal dust-cored coil negligible compared with the core eddy-current loss.

31. Best shape for toroidal dust-cores

The best proportions to be used for a dust-core toroid depend to some extent on the permeability of the material, and on the purpose for which the coil is to be used. Experience has shown that the best results are obtained when the core cross-section is in the form of a "rectangle" with rounded corners, the radial thickness varying between 0·18 and 0·25 of the mean radius of the ring. The axial thickness of a single ring is usually about the same as its radial thickness, but where the hysteresis factor of the finished coil must be kept as low as possible, the core volume is sometimes increased by using two or more rings, placed side by side, and taped together to form a single core. The d.c. resistance factor and the other core loss factors are not affected very much by increasing the number of rings. The relatively large radial thicknesses are found in low-permeability cores, designed to give low hysteresis factors, while the cores used at low frequencies, where the primary requirement is a low d.c. resistance factor, are "thin" and made of high-permeability material.

32. Coils of other shapes

Although it is difficult to obtain exact relationships between the loss factors and the material loss coefficients, for toroids whose windings are not uniformly distributed or for non-toroidal shapes generally, it is sometimes possible to make a rough estimate of the performance to be expected when a core of a given size and shape is made from a dust-core material whose properties are known from tests on a toroidal sample of the same material. Quite apart from the difficulty of calculation, such estimates are often found to be misleading, because, although it is fairly safe to assume that the material of a toroidal core is homogeneous, the varying forming pressures which occur during the manufacture of a more complicated shape, such as a "pot" core, may lead to internal stresses and variations in magnetic properties from point to point within the core. In spite of the difficulties involved, however, it is useful to be able to predict the order of the quantities concerned, even if there is a possibility of errors as high as two to one in their values.

(a) POWER LOSS PER UNIT VOLUME OF CORE

An expression for the hysteresis loss per unit volume of a dust-core can be obtained from (121),

$$\delta P_h = \frac{a_d f}{3\pi} \hat{H}^3 \,.\, 10^{-7} \text{ watt-secs./cu. cm.}$$

$$-\frac{hf\hat{B}^3}{8\pi}\cdot 10^{-7}$$

$$\frac{hfB^3}{2\sqrt{2}\pi}\cdot 10^{-7} \quad \ldots\ldots \quad (172)$$

The eddy-current loss in a toroidal core is R_eI^2, where

$$R_e = \mu_d e f^2 L$$

$$\text{but } L = \frac{4\pi N^2 A \mu_d}{l}\cdot 10^{-9} \quad \ldots\ldots \quad (173)$$

$$\text{and } H = \frac{4\pi NI}{10l} \quad \ldots\ldots \quad (174)$$

$$\text{or } N = \frac{10lH}{4\pi I} \quad \ldots\ldots \quad (175)$$

substituting for L,

$$R_e = \frac{4\pi\mu_d{}^2 A N^2 e f^2}{l}\cdot 10^{-9} \quad \ldots\ldots \quad (176)$$

so that, from (175),

$$R_eI^2 = \frac{1}{4\pi}\mu_d{}^2 A l e f^2 H^2\cdot 10^{-7}$$

$$= \frac{1}{4\pi}B^2 A l e f^2\cdot 10^{-7} \quad \ldots\ldots \quad (177)$$

i.e., the power loss per unit volume is $\delta P e$, where

$$\delta P_e = \frac{eB^2f^2}{4\pi}\cdot 10^{-7} \quad \ldots\ldots \quad (178)$$

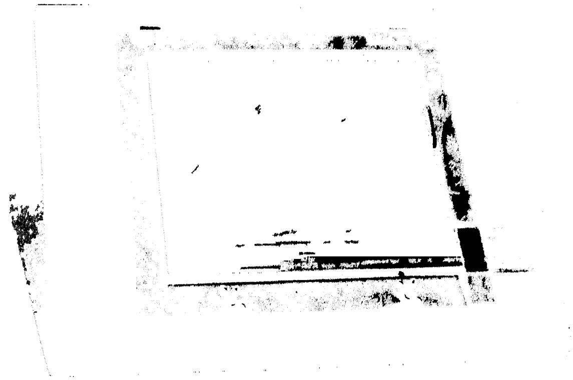
In the same way, the residual power loss can be shown to be

$$\delta P_c = \frac{cB^2f}{4\pi}\cdot 10^{-7} \quad \ldots\ldots \quad (179)$$

The total power loss in a core of any shape could be obtained by imagining the core to be made up of small elements of volume, obtaining an expression for the loss for each element, and then integrating over the whole core. The loss factors would then be given by

$$F_h = \frac{\Sigma\delta P_h}{I^3 f L} = \left(\frac{B}{I\sqrt{L}}\right)^3\cdot\frac{h}{2\sqrt{2}\pi}\cdot 10^{-7} \quad \ldots \quad (180)$$

Plate VII



Courtesy G P O Engineering Dept

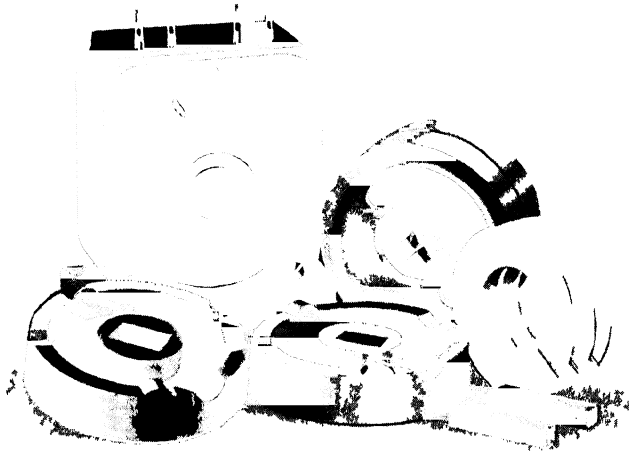
Photograph of Apparatus described in para 51

Plate VIII

Courtesy G P O Engineering Dept

Examples of non Toroidal Dust-cores

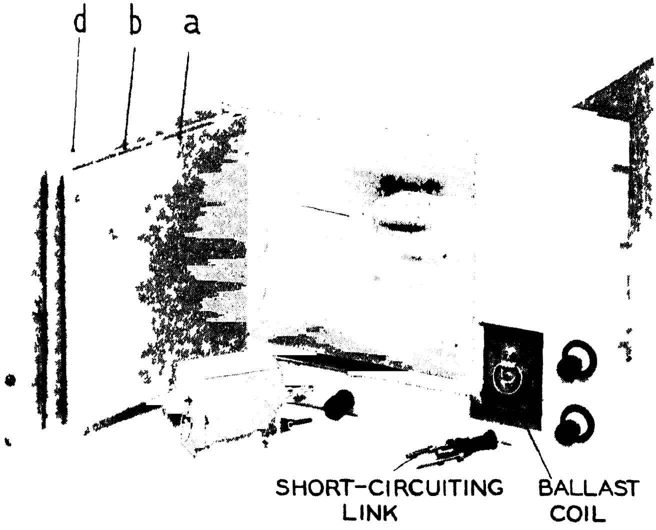
PLATE IX



Courtesy the Telephone Manufacturing Co, Ltd

Construction of a Pot type Dust cored Coil

PLATE X



Courtesy the Post Office Electrical Engineers Journal

Example of Screening used in Impedance measuring Equipment (See Fig 51.)

$$F_e = \frac{\Sigma\delta P_e}{I^2 f^2 L} = \left(\frac{B}{I\sqrt{L}}\right)^2 \frac{e}{4\pi} \,.\, 10^{-7} \quad . \quad . \quad . \quad (181)$$

$$F_c = \frac{\Sigma\delta P_c}{I^2 f L} = \left(\frac{B}{I\sqrt{L}}\right)^2 \frac{c}{4\pi} \,.\, 10^{-7} \quad . \quad . \quad . \quad (182)$$

It is thus necessary to know the value of the quantity $\left(\frac{B}{I\sqrt{L}}\right)$ for each element of volume in order to be able to calculate the core loss factors. One way of avoiding the tedious computation which would be required would be to arrange for a toroidal sample to be made of the same material as the core under consideration. The toroid can then be used to measure the loss coefficients of the material, which can be compared directly with the loss factors of the coil wound on the other core, thus establishing the relationships between coefficients and factors for that particular core shape and for that *particular material permeability*. It must be understood that, even if such a relationship is found for a certain material permeability, it does not follow that the same relationship will hold good if the core permeability is changed. For example, if the core could be replaced with a material with exactly the same values of e, c and h, but which had just twice the original permeability, the core loss factors would not necessarily be doubled. This would occur only in the particular case of a uniformly-wound toroid, and, for any other type of coil, the change of permeability would result in a change of flux distribution, and therefore a change in the values of the loss factors. This point has been stressed because, if it is not fully understood, serious errors may be made in attempting to apply results, obtained for a sample core, to the design of coils on cores of the same type, but of different core materials.

(*b*) Loss Factors of Non-toroidal Coils

Fig. 26 is a diagram representing a generalized dust-cored coil, and showing the possible types of magnetic flux paths which may exist. These are

(*a*) wholly in air
(*b*) partially in air and partially in the core
(*c*) wholly in the core.

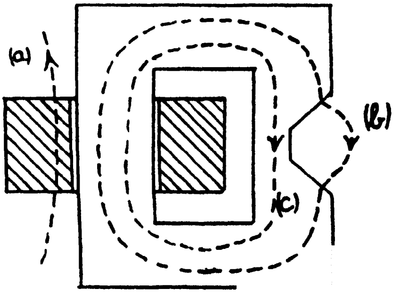
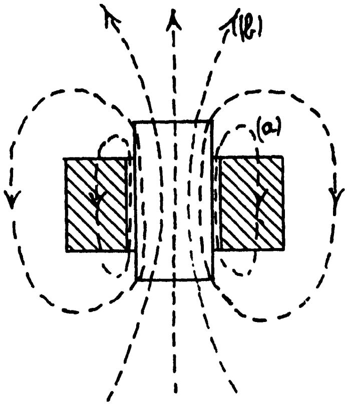
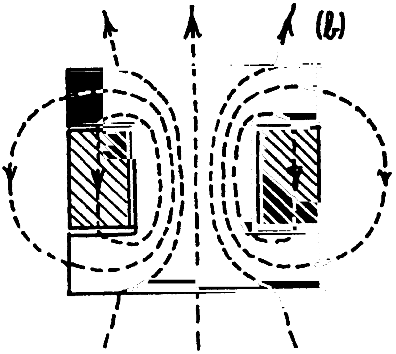


Fig. 26. Diagram representing the cross-section of a generalized dust-cored coil.

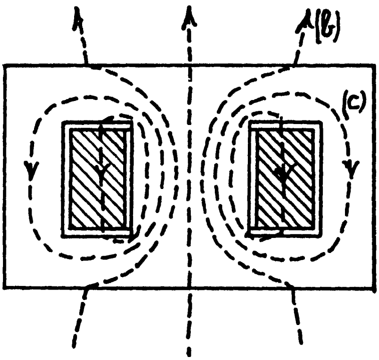
The whole coil will behave somewhat as though it were made up



"Plug" Core.



"Cotton-reel" Core.



"Pot" Core.

Fig. 27. Application of Fig. 26 to some typical dust-core shapes.

of three coils connected in series, one completely air-cored, one with an ungapped dust-core, and the third with a gapped dust-core. Its behaviour will naturally depend on the relative importance of these three components. Fig. 27 shows several possible arrangements of core and winding, indicating roughly the form of the magnetic field in each case. The letters (*a*), (*b*) and (*c*) refer to the flux paths mentioned above.

A simple type of dust-cored coil, often used at frequencies of several megacycles per second, consists merely of a solenoidal winding with a cylindrical plug of dust-core material inserted inside. The plug is sometimes moulded with a screw thread, or is attached to a screwed shank, so that small adjustments can be made to the inductance of the coil by moving the plug axially inside the winding. It is found experimentally that if the length of the plug does not differ greatly from that of the winding, the core loss factors are given, very roughly, by

$$\mathrm{F_e}=\left(\frac{m-1}{n}\right)e \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (183)$$

$$\mathrm{F_c}=\left(\frac{m-1}{n}\right)c \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (184)$$

$$\mathrm{F_h}=\left(\frac{m-1}{n}\right)' h\sqrt{\frac{8\pi}{\mathrm{V}}}\,.\,10^{3.5} \quad .. \quad . \quad . \quad (185)$$

where *m* is the " inductance ratio " (i.e., the ratio between the inductance of the coil with and without the core), *n* is a numerical factor roughly equal to unity, and V is the volume of the core. For estimating purposes it is sufficient to assume that the winding eddy-current loss is not affected by the core, so that

$$k_{\mathrm{d}}=\frac{k}{m}\ \text{(approx.)} \quad . \quad . \quad . \quad . \quad . \quad . \quad (186)$$

If the plug core is extended outside the winding, by adding a disc at each end, a " cotton-reel " shape is obtained and the above expressions will also apply to cores of these types, provided the appropriate values of *n* and V are used. The values of both these quantities depend on the distribution of flux density throughout the core ; for example, if the flux density remained practically uniform in the part of the core enclosed by the winding and then fell off rapidly outside the winding, the value of *n* would be about unity and V would be the volume of core enclosed by the winding. Experiment shows that *n* is about 0·7 for a cotton-reel core, and varies for pot cores between 0·3 for a core with its overall length about twice its overall radius and with a relatively thin outer shell, and 0·7 for a short core with a thick outer shell. For estimating purposes, V can be taken as equal to the volume enclosed by the winding in each case. This is sufficiently accurate because, in dealing with hysteresis, it is always found that the performance of non-toroidal dust-cores is liable to vary considerably from sample to sample, presumably owing to lack of uniformity in the structure of the core. The main advantage of the extra material used in the cotton-reel and pot cores is to reduce the external field of the coil and therefore to enable the screening-can to fit more closely around the coil. At very high frequencies, this advantage is outweighed by the increased self capacitance and by losses due to the electric field, which cause circulating currents to flow in the core *via* the capacitances between the particles of material of which it is made. This effect occurs even in parts of the core where the magnetic flux density is low, and the loss increases rapidly with frequency. Although it is impossible to state any hard-and-fast rules, it will be found that the required inductance ratios of dust-cored coils increases as the working frequency is reduced, the best type of coil ranging from a solenoid with a plug core having $\mu_d = 12$ and $m = 2$, for use at a frequency of several megacycles per sec., down to a toroid with $\mu_d = 125$ and $m = 60$ for use at about 1000 cycles per sec. At frequencies lower than this, where higher inductance ratios are required, better results will probably be obtained by changing over to cores with gapped laminations or, alternatively, to a core composed of one of the ferrite materials mentioned in § 34.

33. Comparison of toroids and pot cores

In order to compare the properties of toroidal and pot-type dust cores, we will consider two cores, one of each type, made of the same material and arranged so that the mean radius r_0 of the toroid and the outer radius r_1 of the solenoidal winding of the pot core are equal and denoted by r. The toroid has $\frac{a}{r_0} = 0{\cdot}25$, and the winding of the pot core has $\frac{r_1}{l} = 1{\cdot}4$ and $\frac{t}{r_1} = 0{\cdot}33$. The winding space is assumed to be filled in each case. The value of γ is obtained, from Figs. 11 and 10, as 7000 and 2700, respectively, for the toroid and the solenoid.

The d.c. resistance factors are thus

$$K_{d_1} = \frac{7000}{\lambda r^2 m_1} \text{ (for the toroid)}$$

$$K_{d_2} = \frac{2700}{\lambda r^2 m_2} \text{ (for the pot core)}$$

where m_1 and m_2 are the appropriate inductance ratios. The core eddy-current factors are, respectively,

$$F_{e_1} = e(m_1 - 1)$$

$$F_{e_2} = \frac{e(m_2 - 1)}{n} \text{ (approx.)}$$

Now, assuming that the strand size used in the winding is sufficiently small to make the winding eddy-current loss negligible, the peak Q (neglecting residual loss) is inversely proportional to $\sqrt{F_e k_d}$. The peak Qs for the two types of coil are then in the ratio

$$\sqrt{\frac{2700}{7000}\left(\frac{m_1}{m_1 - 1}\right)\left(\frac{m_2 - 1}{m_2 n}\right)}$$

For the shape chosen, n will be about 0·7, so that if $m_2 = 6$, the ratio will be about 2 : 3, the pot coil having the higher value of Q. The ratio between the peak Q's of the two coils will be slightly reduced when residual loss is taken into account, and further reduced by the fact that, as a rule, even with the smallest practicable strand size, the winding eddy-current loss in the pot core cannot be made quite negligible. The screening-can of the toroid can be made to fit closely over the coil but space must be allowed for the pot coil, not only to accommodate the clamps which hold the parts of the core together, but also to prevent undue screen loss. As a result, the total space occupied by the pot core will be somewhat greater than that needed for the toroid ; conversely, for equal volume occupied, there will be very little difference between the peak Q_0 values for the two types of coil.

Finally, we must compare the hysteresis factors of the two coils when arranged to give identical Q_0 frequency curves. The volume of the toroid is

$$2\pi^2 a^2 r_0 = 2\pi^2 \left(\frac{a}{r_0}\right)^2 r_0^3$$

$$= 1{\cdot}23 r_0^3,$$

and the effective volume of the pot core is the volume enclosed by the winding, i.e., $\pi r_1^2 l$

$$= \pi \left(\frac{l}{r_1}\right) r_1^3$$

$$= 2{\cdot}25\, r_1^3.$$

Since in the example $r_0 = r_1$, the effective volume of the pot core is about 1·8 times that of the toroid. For equal eddy-current factors,

$$(m_1 - 1) = \left(\frac{m_2 - 1}{n}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (187)$$

and the toroid will therefore have a hysteresis factor about 1·35 times that of the pot core. To satisfy (187), the material of the toroid will need to have a smaller packing factor, and, although theoretically this should not affect the value of the hysteresis coefficient h, it is found in practice that dust-core materials with low permeabilities tend to have high values of h. To put this another way, where there is a possible choice between a reduction of packing factor and the introduction of an effective air-gap (as in a pot core), lower hysteresis factors will probably be obtained by the latter method. We have now reached the conclusion that, for a given volume, the pot core gives a Q_0 at least as high as that obtainable with a toroid; it has a better hysteresis factor; and is, of course, easier to wind and to adjust. The only limitation of the pot core is in the inductance ratio, and it is at relatively low frequencies, where the highest possible value of m is desirable, that the advantage of the toroid becomes apparent. It is difficult to lay down any definite rules on this point, but it can be stated that the useful frequency range of pot type dust-cores extends from about 50 kc/sec. to 500 kc/sec., while toroids are preferable over the lower part of the frequency range for which dust-core coils are suitable (say 5 to 50 kc/sec.).

34. Ferrite cores

Although this chapter deals mainly with cores made of compressed magnetic powders, mention must be made of " ferrite " cores, which have somewhat similar properties. The ferrites are crystalline double oxides of iron and some other metal (e.g., zinc ferrite, $ZnFe_2O_4$). The general characteristic of these materials is that their specific resistance is so high that they can be regarded as insulators, and yet they exhibit ferro-magnetism, some having quite high permeabilities. Even better results can be obtained with complex crystals* formed by combining two ferrites, and permeabilities of several thousand are not unusual. Since the resistivity is high, eddy-current losses are practically negligible, even when the core is in a solid mass without any attempt at lamination. To all intents and purposes, the eddy-current coefficient of a ferrite material is zero. The residual and hysteresis coefficients are comparable in magnitude to those of a dust core and the ferrite behaves, in fact, like a dust core made up of infinitely small particles and yet possessing permeabilities much higher than can at present be obtained with a dust core.

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* Suitable complex ferrites of this type are marketed under the trade name of " Ferroxcube ", a name suggested by the " cubic " crystal structure of these materials.

CHAPTER VII

SELF-CAPACITANCE OF COILS

For most purposes, the effect of capacitance between the turns of the winding can be considered as a lumped "self-capacitance" shunted across the terminals of the winding. It must be remembered, however, that this is only true if each turn is perfectly coupled to all the other turns. If the coupling is poor (e.g. in a single-layer air-cored coil with the turns spaced well apart) the theory becomes much more complicated. In this book, the self-capacitance will be assumed to be lumped across the coil terminals.

35. Solenoids

It can be shown that, for a given type of wire, the self-capacitance of a single-section multi-layer solenoidal coil is proportional to $l\left(\frac{r_1+r_2}{r_1-r_2}\right)$, where l is the length of the solenoid and r_1 and r_2 are its outer and inner radii respectively. The capacitance will also be a function of the space between the turns, and will obviously increase as the latter are packed more tightly together. When the turns are packed as closely as possible, the self-capacitance is found to be very nearly inversely proportional to $\log\left(\frac{\mathrm{D}}{d}\right)$ for solid wire, or to $\log\left(\frac{\mathrm{D}}{\mathrm{D}_1}\right)$ for stranded wire.

The above results apply to an ideal winding, with the turns wound absolutely uniformly, and wide discrepancies may easily occur in practice. For example, there is sometimes a tendency for turns at the edges of the outer layers to " bed down " until they are in close proximity to turns in lower layers. This will cause sudden changes in the potential gradient along the winding and will increase the self-capacitance. These results, however, do provide a rough guide to the way in which the design of a coil should be modified in order to obtain a desired change in the capacity.

The first thing to be noted is the somewhat unexpected fact that the capacitance depends on the ratio of the inner and outer radii of the coil, and not on their absolute magnitude. In other words, if the inner radius is kept constant, the capacitance decreases as the outer radius is increased. Thus, for example, the capacitance of a solenoidal bobbin is always least when the bobbin is filled with wire. Conversely, for a given outside radius, it is at a minimum when the inner radius is zero. The self-capacitance is directly proportional to the length of the coil, thus leading to the conclusion that, for a given space, the type of solenoid which will have the least self-capacitance is a flat " disc" winding, extending practically to the centre of the coil.

36. Multi-section coils

The self-capacitance of a coil can be reduced by splitting the winding into a number of sections, all connected in series. Suppose that a given solenoidal winding has a capacitance C and that it is subdivided into n sections. Each section will then have $1/n$ of the capacitance of the original coil, and since there are n sections, the total effect is to reduce the capacitance across the coil in the ratio n^2 to 1. At the same time there is the capacitance between pairs of adjacent sections to be taken into account, so that the effective capacitance of the coil is changed from C to

$$\frac{C}{n^2}+\frac{C_1}{n-1},$$

where C_1 is the capacitance between two adjacent sections. It will be noticed that, for small values of n, the capacitance of the sectionalized coil will be inversely proportional to n^2, and will depend on the capacitance of the original single-section coil. As n is increased, the inter-section capacitance becomes more and more important, and the total capacitance tends to become inversely proportional to n. It can be concluded that the self-capacitance of a coil can be reduced almost indefinitely by splitting it up into enough sections, but that, to obtain the fullest advantage, the inter-section capacitance should be kept negligible. This means that as the number of sections is increased, the spacing between them should also increase. This will cut down the available winding space, with the inevitable result that the d.c. resistance factor will also increase. The choice of the number of sections to be used will thus depend on obtaining a satisfactory compromise between self-capacitance and d.c. resistance, bearing in mind the fact that the use of a large number of sections makes construction more difficult and increases the cost. It should be pointed out that, as the self-capacitance of a coil is reduced by progressive sectionalizing, the influence of other stray capacitances (e.g., capacitance to a screening-can), will become more important.

37. Stray capacitances

With iron-cored coils, the apparent self-capacitance can be considerably increased by the proximity of the core. This effect is most noticeable in a " pot " type dust-cored coil, in which the " core " is extended in the form of a shell which completely encloses the winding (Fig. 28 (*a*)). A circuit will exist through the capacitance between the inner layer of the winding (A) and the core, through the core itself, and back to the outer layer of the winding (B), through the capacitance between the latter and the core. The path through the core consists partly of the capacitance between the particles of iron of which it is composed, and partly of leakage through the insulating layers which separate them. These stray capacitances between wind-

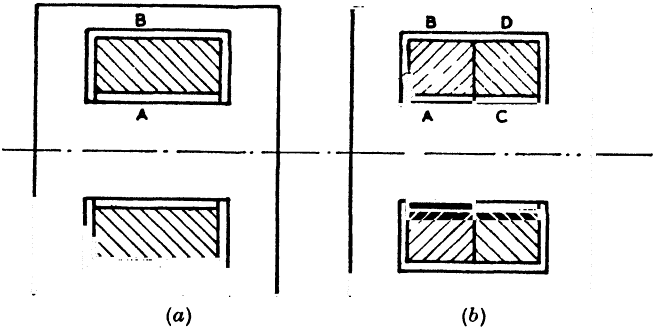


FIG. 28. Diagram illustrating the stray capacitances in a pot type dust-cored coil.

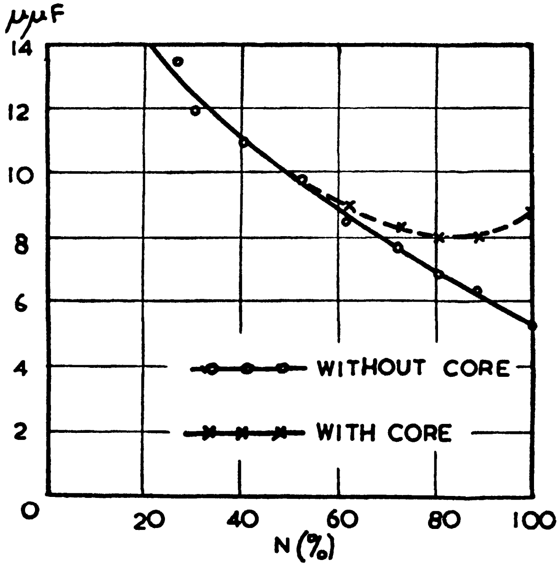


FIG. 29. Measured self-capacitance of a pot type dust-cored coil, showing the effect of the core.

ing and core should be kept small, not only because they increase the apparent self-capacitance of the coil, but also because power is dissipated by the currents which they produce in the core. Since the two stray capacitances are in series, their effect will not be considerable unless both the inner and outer layers of the winding are allowed to approach the core simultaneously. Where self-capacitance is important, coils of this type should not be wound with more than about 80% of the bobbin winding space occupied. This is illustrated in Fig. 29, which shows the measured self-capacitance of a pot type coil, with and without the core in position. If the winding is split into two sections (AB, CD, in Fig. 28 (*b*)) the stray capacitance for each section is reduced approximately to half, so that, since there are two sections in series, the total effect is to reduce it to one-quarter across the whole coil. The capacitance of the winding has been reduced roughly in the same ratio, so that the relative importance of the stray capacitance between winding and core may not be reduced by sectionalizing ; in fact, it may be made worse, since there will now be an additional path, through the central core, between layers A and C, and, through the outer shell, between B and D.

38. Toroidal coils

As the first layer is wound on to a toroidal former, the measured self-capacitance will be found to rise slowly ; then, as the first layer is completed and successive turns begin to lie on top of those at the beginning of the winding, there is a sudden increase in capacitance (see Fig. 30). A corresponding jump in capacitance occurs as each

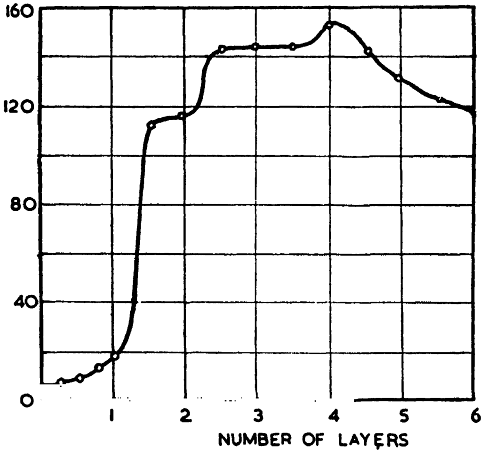


FIG. 30. Measured self-capacitance of a multi-layer toroidal coil.

layer is completed, although these increases become smaller and smaller, since each additional inter-layer capacitance is shunted across a smaller proportion of the whole winding. Finally, a maximum value is reached, and the capacitance then falls off steadily. No attempt has been made to derive a theoretical expression for the capacitance of a toroid, because it has been found that the capacitance is often critically dependent on the exact way in which the wire is wound on to the former. It has been mentioned that " bedding down " of the turns in outer layers can cause a considerable increase in capacitance, and a consideration of the peculiar form of a toroidal winding will show that there is likely to be a great tendency for turns in outer layers to drop down, at the outside of the ring, into gaps between turns in lower layers.

It can be concluded that, from the point of view of self-capacitance, the best toroidal winding is one with a single layer, and with the ends separated. Alternatively, a deep winding with many turns will have a fairly low capacitance.

39. Single-layer solenoids

It has been found experimentally that the self-capacitance of single-layer solenoidal coils depends solely on their overall shape and size, and is independent of the relative spacing of the turns. The self-capacitance, expressed in micro-micro-farads, is of the same order as the coil diameter in centimetres, the actual values being given in Fig. 31 for coils with one end earthed. It is seen that, for a given diameter, the self-capacitance is a minimum when the length of the coil is roughly equal to its diameter. In dealing with coils of this type, where the self-capacitance is itself so small, it should be remembered that the question of stray capacitances, due to leads, screening-can, etc., becomes of great importance, since these can easily swamp the internal self-capacitance between the turns of the coil.

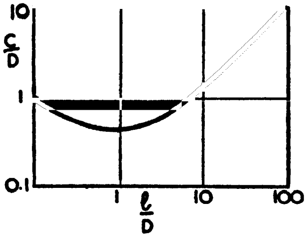


FIG. 31. Curve for estimating the self-capacitance of a single-layer solenoidal coil (after Medhurst). C is the capacitance in $\mu\mu$F.

CHAPTER VIII

Q FREQUENCY CURVES FOR AIR-CORED COILS

40. Variation of Q with frequency

It was shown in Chapter II that, if the impedance of a coil is represented by the equivalent circuit shown in Fig. 2 (*a*), its Q is given by

$$Q = Q_0 \left[1 - x^2 \left(1 - \frac{1}{Q_0^2}\right)\right] \quad . \quad . \quad . \quad . \quad (188)$$

$$\text{where } Q_0 = \frac{\omega L}{R} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (189)$$

$$\text{and } x^2 = \omega^2 LC \quad . \quad . \quad . \quad . \quad . \quad . \quad (190)$$

It can also be shown that, making certain assumptions, the values of L and C can be taken as independent of frequency, whilst the resistance R can be expressed in the form

$$R = L\,(K + kf^2) \quad . \quad . \quad . \quad . \quad . \quad (191)$$

where K and k are constants whose values depend on the shape and dimensions of the coil, and on the winding details. It will be noted that if $C = O$, $Q = Q_0$. Thus Q_0 can be regarded as the " ideal " value which Q would have if the coil had no self-capacitance. The expression $\left[1 - x^2 \left(1 - \frac{1}{Q_0^2}\right)\right]$ is then a correction factor which takes the self-capacitance into account.

(191) can now be used to express Q_0 as a function of frequency

$$Q_0 = \frac{\omega L}{R}$$

$$= \frac{2\pi f L}{L\,(K + kf^2)}$$

$$= \frac{2\pi}{\frac{K}{f} + kf} \quad . \quad . \quad . \quad . \quad . \quad . \quad (192)$$

It is clear from the form of this equation that there will be a finite value of f for which Q_0 is a maximum. The condition for this to occur is $\dfrac{dQ_0}{df}=0$, which reduces to $kf=\dfrac{K}{f}$.

Denoting the maximum value of Q_0 by $\hat{Q}_0$, and the frequency at which it occurs by f_0, we get

$$\hat{Q}_0=\frac{\pi}{\sqrt{Kk}} \qquad (193)$$

$$\text{and } f_0=\sqrt{\frac{K}{k}} \qquad (194)$$

f_0 and $\hat{Q}_0$ are the co-ordinates of the peak of a curve of Q_0 plotted against f. The value of Q_0 at any frequency f, can be expressed in terms of $\hat{Q}_0$ and f_0 in the following way. Divide the numerator and denominator of the right-hand side of (192) by $\sqrt{Kk}$,

$$Q_0=\frac{\dfrac{2\pi}{\sqrt{Kk}}}{\dfrac{1}{f}\sqrt{\dfrac{k}{K}}+f\sqrt{\dfrac{K}{k}}} \qquad (195)$$

which, making use of (193) and (194), becomes

$$Q_0=\frac{2\hat{Q}_0}{\dfrac{f}{f_0}+\dfrac{f_0}{f}} \qquad (196)$$

Finally, writing $\dfrac{f}{f_0}=y$, we obtain the equation

$$Q_0=\frac{2\hat{Q}_0}{y+\dfrac{1}{y}} \qquad (197)$$

which can be rearranged in the form

$$\frac{Q_0}{\hat{Q}_0}=\frac{2}{y+\dfrac{1}{y}} \qquad (198)$$

It will be noticed that the right-hand side of this equation is unaltered if y is replaced by $\dfrac{1}{y}$. This means that if $\log\dfrac{Q_0}{\hat{Q}_0}$ is plotted against $\log y$, then since $\log y=-\log\dfrac{1}{y}$, it follows that the graph must be symmetrical about the line $y=1$. This is illustrated in Fig. 32, which shows the form taken by the curve of $\dfrac{Q_0}{\hat{Q}_0}$ plotted against y, using logarithmic scales on both axes. In this way a general curve is obtained, for which the co-ordinates of the peak are both unity.

Suppose now that the Q_0 frequency curve is required for an air-cored coil whose K and k constants are known. It is a simple matter to obtain the values of $\hat{Q}_0$ and f_0 from (193) and (194) and to use these to plot the peak of the curve on log.-log. graph paper, with the axes calibrated in Q_0 and frequency units. A tracing of the curve of Fig. 32, to the same scale, can then be located with its peak coinciding with the plotted point, and used as a template to plot the required Q_0 frequency curve.

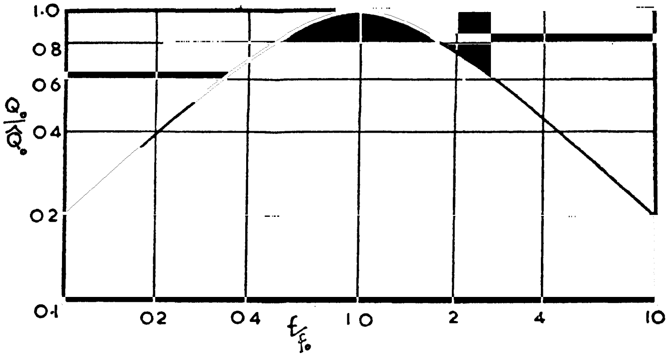


FIG. 32. Q-frequency curve for any air-cored coil in which self-capacitance and eddy-current shielding can be neglected.

(a) EFFECT OF SELF-CAPACITANCE OF COIL

It will be seen from (188) that, although the self-capacitance correction factor is, strictly speaking, a function of Q_0 as well as of x, yet, for all reasonable values of Q_0, the term $\frac{1}{Q_0^2}$ can be neglected in comparison with unity. We then obtain the simplified equation

$$Q = Q_0 (1 - x^2) \quad . \quad . \quad . \quad . \quad . \quad . \quad (199)$$

Before this correction factor can be applied to the general curve of Fig. 32, a further step is necessary. This is to express x in terms of the frequency f_0, at which point the peak of the Q_0 frequency curve occurs.

By definition $x = \omega\sqrt{LC}$

$$\text{or } x = \frac{f}{f_r} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (200)$$

$$\text{where } f_r = \frac{1}{2\pi\sqrt{LC}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (201)$$

Then, since $y = \frac{f}{f_0}$

we have $x = y\left(\frac{f_0}{f_r}\right)$ . . . . (202)

So that (199) can be written in the form

$$Q = Q_0\left[1 - y^2\left(\frac{f_0}{f_r}\right)^2\right] \qquad . \quad . \quad . \quad . \quad . \quad (203)$$

Fig. 33 shows curves of $\frac{Q}{Q_0}$ plotted against y for various values of $\frac{f_0}{f_r}$. The family of curves given in Fig. 33 now enable the Q frequency curve (i.e., including the effect of self-capacitance), to be drawn for any air-cored coil for which the values of K, k and f_r are known.

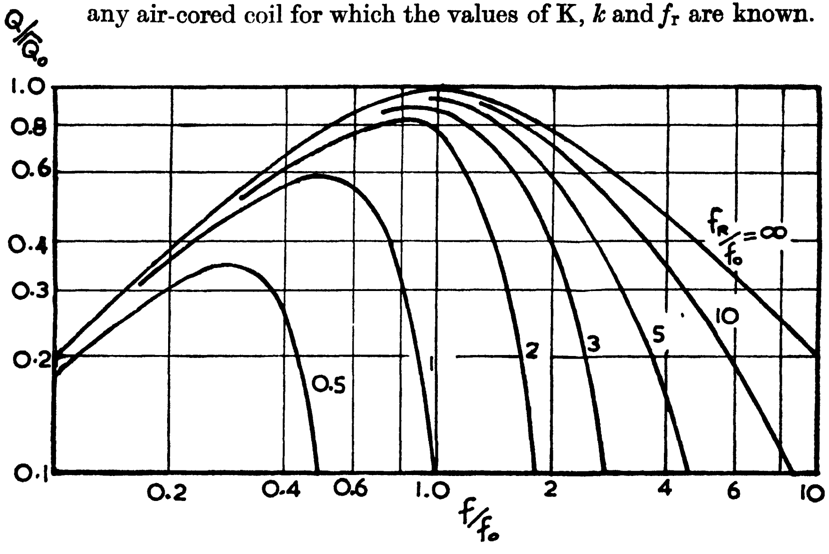


FIG. 33. Modification of Fig. 32 to take into account the self-capacitance of the coil.

(b) RESONANT FREQUENCY OF COIL

Care should be taken in interpreting the significance of the term $f_r = \frac{1}{2\pi\sqrt{LC}}$, which is introduced merely for convenience. f_r does not necessarily coincide with the " natural " or " self-resonant " frequency of the coil; i.e., the frequency at which the impedance of the coil becomes a pure resistance. So far we have assumed that L and C are independent of frequency, an assumption which can be justified only over a limited frequency range, and which may no longer apply at the frequency corresponding to f_r. It is clear that, in such a case, the impedance of the coil would not be purely resistive at this frequency. It must be stressed therefore that (203) can be applied only over the working frequency range of the coil, where L and C, the components of the equivalent circuit, can be assumed to be constants.

41. Variation of Q_0 with the dimensions and winding details of multilayer coils

The results obtained above can be used, in conjunction with the expressions, given in § 10 (*f*) (p. 37), for *k* and K, in order to reach certain conclusions about the behaviour of Q_0 as the dimensions of the winding are changed.

Examination of (51) and (60) shows that, for coils of identical proportions, but varying dimensions,

$$K \propto \frac{1}{[L]^2} \cdot \frac{1}{\lambda} \qquad (204)$$

$$\text{and } k \propto \lambda d^2 \qquad (205)$$

where [L] represents the dimension of length, so that, from (193) and (194)

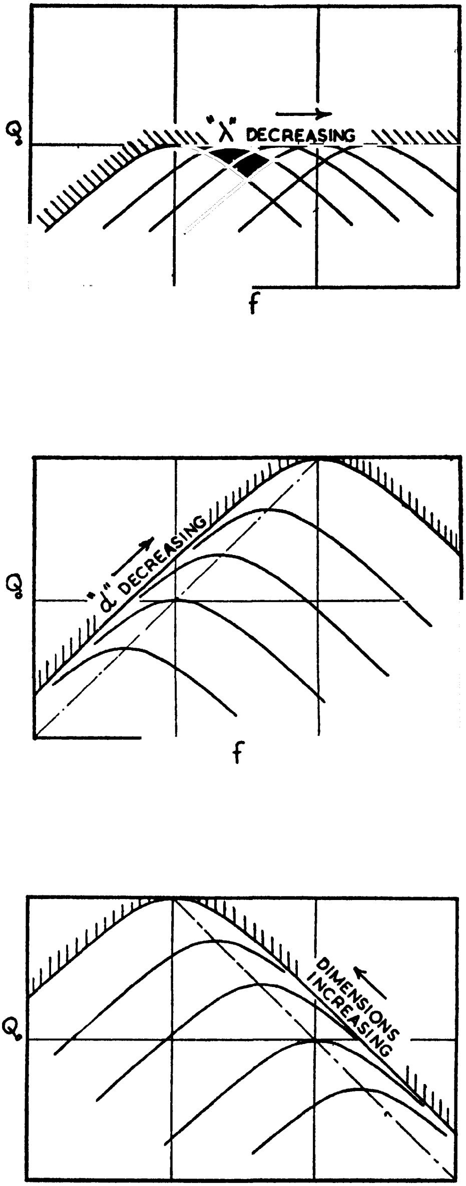
$$f_0 \propto \frac{1}{d\lambda[L]} \qquad (206)$$

$$\hat{Q}_0 \propto \frac{[L]}{d} \qquad (207)$$

where *d* is the wire diameter (or strand diameter in the case of stranded wire), and λ is a space factor which defines, in effect, the proportion of the total winding volume which is actually occupied by metallic conductors. Thus, for multilayer coils of identical shape, we are led to the following important conclusions :

(i) Both $\hat{Q}_0$ and f_0 are independent of the inductance of the coil.
(ii) $\hat{Q}_0$ is directly proportional to the linear dimensions, and f_0 is inversely proportional to the dimensions.
(iii) Both $\hat{Q}_0$ and f_0 are inversely proportional to the wire or strand diameter.
(iv) f_0 is inversely proportional to the space factor, but $\hat{Q}_0$ is independent of λ.

These results are illustrated diagrammatically in Fig. 34. In each diagram, the chain line represents the locus of the peaks of the curves. The shaded area in each case is a " forbidden " region for Q_0, the limits being set :



Dimensions and "d" constant.

Dimensions and "λ" constant.

"d" and "λ" constant.

FIG. 34. Diagrams to illustrate the way in which the Q-frequency curves of a multi-layer air-cored coil depend on its dimensions and winding details.

(*a*) by the maximum allowable dimensions for the completed coil, which are dictated by considerations of space occupied, or of cost

(*b*) by mechanical and economic limits to the fineness of the strands

(*c*) by the fact that, for wire of circular cross-section and normal insulation thickness, the value of the space factor cannot exceed about 0·6.

(*a*) Optimum Dimensions

Consideration of these results shows clearly the fact that coils designed to give high Q_0 values at low frequencies will tend to become bulky and expensive, while at higher frequencies the tendency is towards small physical dimensions and finely stranded wire. At first sight Fig. 34 (*a*) seems to suggest that the frequency at which the peak of the Q_0 frequency curve occurs can be increased indefinitely by making λ smaller and smaller. It must be remembered, however, that these results are based on assumptions which are justifiable only over a limited frequency range. As the frequency is increased, a limit will be reached where the skin effect in the wire can no longer be neglected, because it is not practicable to make the strand diameter sufficiently small, and the present simple theory becomes invalid. At frequencies above this limit (say about 10 Mc./s.), it is usually found preferable to abandon stranded wire altogether in favour of heavy solid wire or tube, the coil consisting of a few turns, spaced well apart (see § 44).

(*b*) Choice of Wire

There is an interesting application of the above results, which is of importance in the economical design of air-cored filter coils, wound

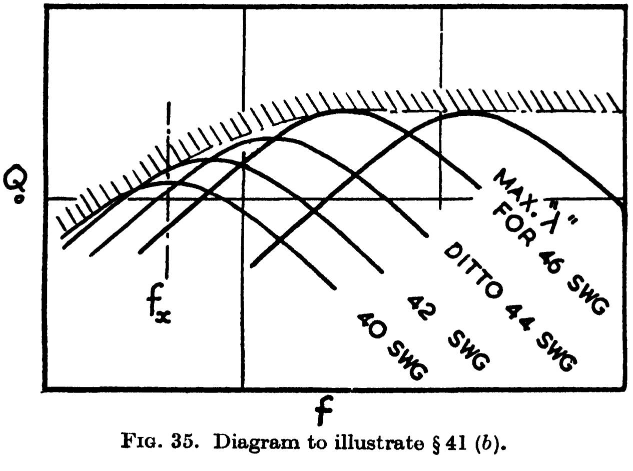


Fig. 35. Diagram to illustrate § 41 (*b*).

on standard bobbins. Suppose that the shape and dimensions of the winding are all fixed, but that the wire details are changed, i.e., λ and d are both variables. At first it appears that the highest value of Q_0 will be obtained at all frequencies by using the smallest possible strand diameter. Very finely stranded wire, however, tends to have a poor space factor, owing to the increased relative thickness of the insulation, so that the maximum value of λ will decrease as d is decreased. The result of this is illustrated in Fig. 35, which shows the form taken by the curves for a hypothetical air-cored inductor. It will be seen that, at a frequency such as f_x, a higher Q_0 value is obtained with 42 S.W.G. strands than with the limiting size of 46 S.W.G. (i.e., the minimum practicable size of strand in this particular case, for the reasons already stated). An analysis of this kind often enables the unnecessary use of relatively expensive finely stranded wire to be avoided.

42. Influence of winding shape on the Q_0 frequency curves of multi-layer solenoidal coils

The loss factors of a solenoidal air-cored coil can be expressed in the following form

$$K = \frac{\gamma}{r_1{}^2\lambda} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (208)$$

$$k = \beta\lambda d^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (209)$$

where r_1 is the external radius of the winding (see Fig. 3) and γ and β are " shape factors ". Subject to the assumptions made at the beginning of the chapter, γ and β will be quantities whose values are independent of the actual size of the coil, or of the number of turns of wire, and are determined only by the shape ; i.e., by the *relative* values of the dimensions shown in Fig. 3.

Substituting the above expressions for K and k in (193) and (194) we obtain

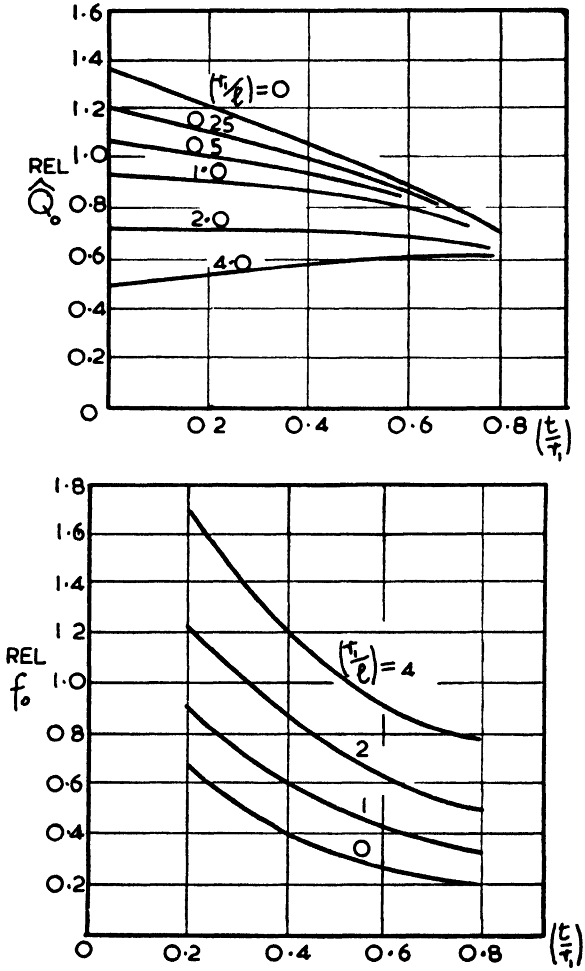
$$f_0 = \frac{1}{r_1 d\lambda}\sqrt{\frac{\gamma}{\beta}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (210)$$

$$\hat{Q}_0 = \frac{\pi r_1}{d\sqrt{\gamma\beta}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (211)$$

These equations finally define the Q_0 frequency curve in terms of the external radius of the winding, the wire specification and the shape factors γ and β, whose values have been investigated in Chapter III. We can now proceed to apply the results of Chapter III to the problem of obtaining the best shape and size of coil to give any required Q_0 frequency curve.

COILS WITH FIXED EXTERNAL RADIUS

The choice of the external radius to define the size of the coil is considered to be a logical one, because the problem in practice often resolves itself into that of designing the most efficient coil which will fit into a certain space.



FIGS. 36 and 37. Curves showing the way in which the performance of a multilayer air-cored solenoidal coil depend on its shape. (Self-capacitance and eddy-current losses neglected.)

Suppose, then, that the wire specification is kept constant, thus fixing the values of λ and d, and suppose also that the external radius, r_1, of the winding is fixed. Fig. 36 shows curves of *relative* values of

$\hat{Q}_0$, plotted against $\left(\frac{t}{r_1}\right)$ and $\left(\frac{r_1}{l}\right)$, where t and l are respectively the thickness and length of the winding as defined in Fig. 3. The following conclusions can be drawn from a study of these curves :

(i) For any given external radius, the highest value of $\hat{Q}_0$ is obtained with l large and t small ; i.e., with a very long solenoid of shallow winding depth.

(ii) For any given external radius and a particular proportion, say $\left(\frac{r_1}{l}\right)=2$, the value of $\hat{Q}_0$ is roughly independent of the winding depth.

(iii) A reasonable practical compromise is to make $\left(\frac{r_1}{l}\right)=0{\cdot}5$, in which case $\hat{Q}_0$ will reach over 80% of the maximum attainable value for the given external radius, provided the winding depth is not too great ; i.e., if $\left(\frac{t}{r_1}\right)$ does not exceed a value of about 0·25.

In the same way, curves of *relative* values of f_0 can be plotted, giving the results shown in Fig. 37. From these we may conclude that :

(i) Where a high value of f_0 is required, the best shape, for a given external radius, will be a short solenoid of small winding depth. (It must be remembered, of course, that f_0 cannot be increased indefinitely, because the whole of the present theory is based on assumptions which apply only over a limited frequency range.

(ii) There is a lower limit to f_0, which is reached as $\left(\frac{r_1}{l}\right)$ approaches zero and $\left(\frac{t}{r_1}\right)$ approaches unity ; i.e., for a long "solid" solenoid in which the winding extends practically to the centre.

43. Toroidal coils

The proportions of a toroidal coil are defined by the values of the ratios $\left(\frac{t}{a}\right)$ and $\left(\frac{a}{r_0}\right)$. Suppose that the mean radius r_0 is fixed, and both a and t are varied, with the object of determining the optimum proportions to give the highest possible Q_0 for a given wire size and space factor. Since $\hat{Q}_0=\frac{\pi}{\sqrt{Kk}}$, $\hat{Q}_0$ is proportional to $\frac{1}{\sqrt{\gamma\beta}}$, and the optimum condition will be reached when the product $(\gamma\beta)$ is a minimum.

A study of Fig. 11 shows that, in order to keep the d.c. resistance factor low, $\left(\frac{t}{a}\right)$ should be as large as possible ; in other words, whatever the value chosen for $\left(\frac{a}{r_0}\right)$, the available winding space should be completely filled. Under these conditions, β is given by the dotted locus curve in Fig. 13, and is seen to pass through a minimum when $\left(\frac{a}{r_0}\right)$ is about 0·5. The peak of the curve thus reaches its highest possible value when $\left(\frac{a}{r_0}\right)$ is about 0·5 and the winding space is completely filled with wire, the value of β then being roughly $3{\cdot}0 \times 10^{-3}$. The frequency at which the peak Q occurs is determined by the ratio between the d.c. resistance factor and the eddy-current factor, which are controlled, respectively, by the value of the mean radius of the coil and the strand diameter of the wire. Since the latter is not usually made less than about 46 S.W.G., this sets a limit below which k cannot be reduced, except by spacing the turns in some way, in order to reduce the space factor λ, or by reducing the ratio $\left(\frac{t}{a}\right)$. In either case, the reduction of k is obtained at the expense of an increase in the d.c. resistance factor. We can sum up by stating that :

(i) The best value of $\left(\frac{a}{r_0}\right)$ for any air-cored toroid is about 0·5.
(ii) At low frequencies, the highest peak Q will be obtained with the winding space completely filled.
(iii) At high frequencies the best results may be obtained with the winding space only partially filled.

These conclusions will be illustrated more clearly by some of the examples given in Chapter XIII.

44. Q frequency curves for single-layer solenoids

The results obtained above have all been based on the assumption that the coils concerned have been working in their " low-frequency " region, where eddy-current shielding in the wire can be neglected. Single-layer solenoids, on the other hand, are generally used at relatively high frequencies, where eddy-current shielding has taken charge and the total resistance is roughly proportional to $\sqrt{f}$. Under these conditions, the Q of the coil is given by an expression of the form

$$Q_0 = kr\sqrt{f} \qquad . \quad . \quad . \quad . \quad . \quad . \quad (212)$$

where r is the mean radius of the coil and k is a factor* whose value depends on the length/radius ratio and on the relative spacing of the turns. k is plotted in Fig. 38 for various coil shapes and wire spacing ratio. Examination of these curves leads to the following conclusions relating to the optimum proportions which a coil should have in order to give the highest possible Q at high frequencies:

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* Not to be confused with the eddy-current factor, also denoted by k.

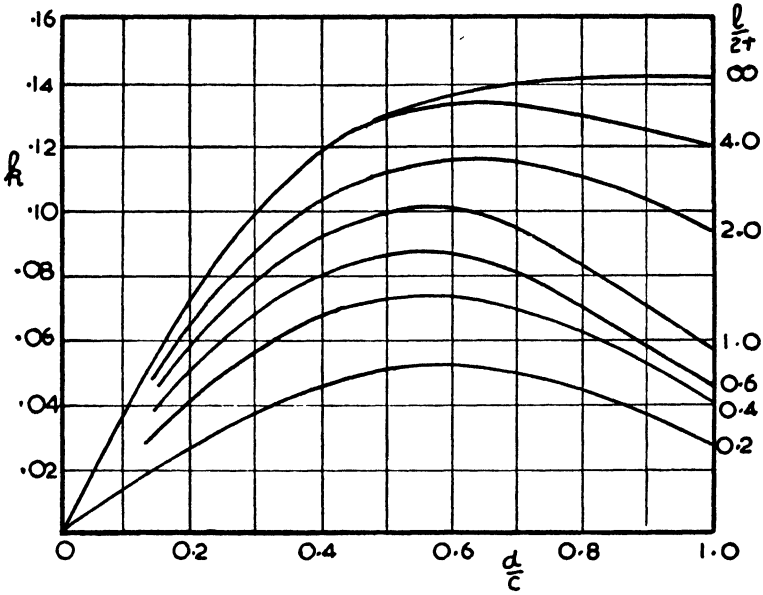

FIG. 38. Curves for calculating the Q of single-layer solenoids at high frequencies; (after Medhurst).

(i) The optimum spacing of the turns, defined by the ratio $\frac{d}{c}$, is about 0·55 for short coils and ranges up to about 0·95 for long coils.

(ii) For a given radius and wire spacing, the maximum Q increases rapidly at first as the coil length is increased, but further improvement is small if the length is made greater than about twice the mean radius.

(iii) A good compromise is to make the length of the coil roughly equal to its diameter, and then to choose the wire diameter to make the ratio $\frac{d}{c}$ about 0·6.

45. Intermediate types

For simplicity, we have so far considered two distinct types of coil, working well above or below the critical "transition" frequency range in which eddy-current shielding begins to become apparent.

An interesting intermediate case can arise, where the frequency f_0, given by (194), falls near the top of the low-frequency region for the type of wire concerned. The Q frequency curve will then take the form indicated by Fig. 39. As the frequency is increased from zero, the Q will at first rise steadily, pass through a maximum at f_0, and then, instead of continuing to fall, will pass through a minimum in the transition region and then start to rise again. Often in such cases it will be found that the self-capacitance of the coil will prevent the use of the latter in the high-frequency region, but let us suppose that the self-resonant frequency is situated as shown in Fig. 39. The Q frequency curve will then pass through a second maximum before finally dropping to zero at the self-resonant frequency. It is thus possible for a coil to be so designed that it has two distinct useful frequency ranges, separated by a range in which the Q falls to a relatively small value. Conversely, it may be found that a given design problem has two possible solutions, one being to use a coil, wound with finely-stranded wire, and working on the lower portion of the curve corresponding to Fig. 39, and the other, to use thick, solid wire, so that the coil works on the upper part of its curve.

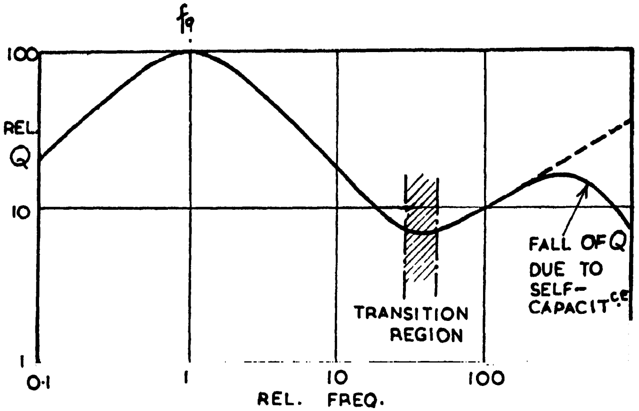

FIG. 39. Diagram showing how a single coil may have two useful frequency ranges.

CHAPTER IX

Q_0 FREQUENCY CURVES FOR IRON-CORED COILS

Let us now consider how the results obtained in the previous chapter will have to be modified when an iron core is introduced. As before, attention will be concentrated on the value of Q_0 ; i.e., the ideal Q which would be obtained if the coil had no self capacitance. A correction can then be made, where necessary, to take the effect of self capacitance into account. Two separate cases will be considered:

(i) dust-cores and laminated cores, working at frequencies where the eddy-current shielding effect in the core can be neglected
(ii) laminated cores, working at frequencies within the eddy-current shielding region.

46. Coils with no eddy-current shielding

For an iron-cored coil, hysteresis and residual losses are introduced which are proportional to frequency, so that another term must be added to (191). The effective series resistance of the coil can be expressed in the form

$$R = (Af^2 + Bf + K)\,L \quad . \quad . \quad . \quad . \quad . \quad (213)$$

$$\text{where } A = (F_e + k) \quad . \quad . \quad . \quad . \quad . \quad . \quad (214)$$

$$\text{and } B = (F_c + F_h I\sqrt{L}) \quad . \quad . \quad . \quad . \quad . \quad (215)$$

Note that K is smaller than the corresponding value for the winding without the core, since the d.c. resistance is unchanged, while the inductance is multiplied by the inductance ratio m, when the core is inserted.

$$\text{Thus } K = \frac{K_a}{m} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (216)$$

(The suffix $_a$ referring to " air-cored " values.)

The relationship between k and k_a is not so simple, because, except for a toroid, the presence of the core will change the intensity of the field cutting the winding, and will therefore change the eddy-current loss for a given number of ampere-turns. In spite of this, since the winding eddy-current loss is usually made small compared with the core losses, the error involved in assuming that

$$k=\frac{k_a}{m} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (217)$$

will not be very great.

This investigation into the Q_0 frequency curves of iron-cored coils will apply mainly to the design of coils for wave-filters, equalizers and other similar apparatus, in which the wave-form distortion due to hysteresis will limit the allowable flux density to a value where the hysteresis loss is also negligible in comparison with the other losses. It is thus reasonable to proceed on the following assumptions :

(i) the strand size of the winding is small enough for the winding eddy-current loss to be negligible
(ii) the hysteresis loss is negligible.

We have already assumed that no eddy-current shielding effects are taking place in the core. Then if, in addition, the insulating materials and the reactance of the coil are so chosen that dielectric losses can be neglected, then

$$R=\left(F_e f^2+F_c f+\frac{K_a}{m}\right)L \quad . \quad . \quad . \quad . \quad (218)$$

which, from (183) and (184), can be written as

$$R=\left[\left(\frac{m-1}{n}\right)ef^2+\left(\frac{m-1}{n}\right)cf+\frac{K_a}{m}\right]L \quad . \quad (219)$$

For estimating purposes, we can take the approximation a step further by assuming $n=1$ and m large compared with 1.

$$\text{then } R=\left(mef^2+mcf+\frac{K_a}{m}\right)L \text{ (approx.)} \quad . \quad . \quad (220)$$

This shows that it should be possible to estimate, very roughly, the probable form of the Q_0 frequency curve of a proposed coil for which we know the inductance ratio m, the loss coefficients e and c of the core material and the d.c. resistance factor of the winding.*

Returning to the general equation

$$R=(Af^2+Bf+K)L$$

$$\text{we have } Q_0=\frac{2\pi fL}{R}=\frac{2\pi}{Af+\frac{K}{f}+B} \quad . \quad . \quad . \quad . \quad (221)$$

Comparison with (192) shows that Q_0 will reach a maximum at a frequency f_0, where

---

* For a laminated core, the value of e is obtained from (107) as $\left(\frac{4\pi^3t^2}{3\rho}\right)$ (approx.), where t is the lamination thickness.

$$f_0=\sqrt{\frac{K}{A}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (222)$$

f_0 will obviously be independent of B, the presence of which merely reduces the value of Q_0 without affecting the frequency at which the peak of the curve occurs. The next step is to express the actual Q_0 frequency curve in terms of the co-ordinates of the peak of a hypothetical curve which would be obtained if B were zero. Denoting the peak value of the latter curve by Q_0', we have

$$\hat{Q}_0'=\frac{2\pi}{\sqrt{AK}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (223)$$

and, dividing numerator and denominator by $\sqrt{AK}$, (221) becomes

$$Q_0=\frac{\dfrac{2\pi}{\sqrt{AK}}}{f\sqrt{\dfrac{A}{K}}+\dfrac{1}{f}\sqrt{\dfrac{K}{A}}+\dfrac{B}{\sqrt{AK}}} \quad . \quad . \quad . \quad . \quad . \quad (224)$$

$$=\frac{2\hat{Q}_0'}{\left(\dfrac{f}{f_0}\right)+\left(\dfrac{f_0}{f}\right)+\dfrac{B}{\sqrt{AK}}} \quad . \quad . \quad . \quad . \quad . \quad (225)$$

$$=\frac{2\hat{Q}_0'}{y+\dfrac{1}{y}+p} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (226)$$

$$\text{and } \hat{Q}_0=\frac{\hat{Q}_0'}{1+\dfrac{p}{2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (227)$$

$$\text{where } y=\frac{f}{f_0} \text{ and } p=\frac{B}{\sqrt{AK}} \quad . \quad . \quad . \quad . \quad . \quad (228)$$

The quantity p defines the relative effect of residual loss on the Q_0 frequency curve. Fig. 40 shows a family of curves of $\dfrac{Q_0}{\hat{Q}_0'}$ against y, plotted for various values of p. The Q_0 frequency curve for a given coil can be obtained from these in the following way:

(i) Calculate f_0 and $\hat{Q}_0'$ from (222) and (223).
(ii) Plot this point on log.-log. graph paper and locate over it the point X on a tracing of Fig. 40 to the same scale.
(iii) Calculate p from (228) and use this to select the correct curve to be used.

Suppose we rewrite (228) in the form

$$p=\frac{B}{\sqrt{AK}}=\frac{F_c\hat{Q}_0'}{2\pi} \quad . \quad . \quad . \quad . \quad . \quad (229)$$

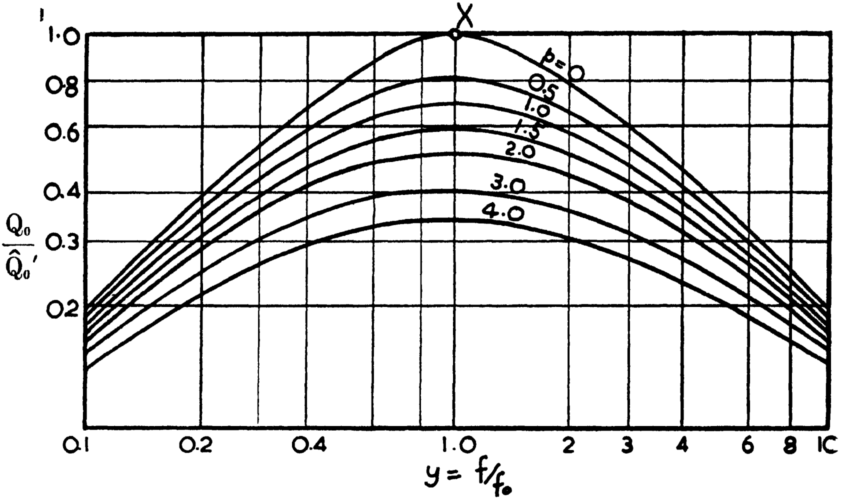

FIG. 40. General Q-frequency curves for an iron-cored coil at frequencies for which eddy current shielding can be neglected.

This shows that, other things being equal, coils with large values of $\hat{Q}_0'$ will tend to have high p values ; i.e., residual loss will have a relatively big effect on the curve. Also, since F_c is independent of the dimensions and $\hat{Q}_0'$ is proportional to the linear dimensions of geometrically similar coils, it follows that p will also be proportional to linear dimensions. The result of this is that, for iron-cored coils, the actual peak Q_0 (i.e., $\hat{Q}_0$) is not necessarily doubled if the linear dimensions are doubled. The peak $\hat{Q}_0'$ of an ideal coil with no residual loss in its core would be doubled but, in a real coil, the increase will be partially counteracted by the increase in the value of p. The Q_0 frequency curve will become " flatter " as the dimensions are increased and the value of the peak Q_0 will depend more critically on p ; i.e., on the residual loss and hysteresis. This effect is particularly noticeable with coils wound on ferrite core materials (see § 34) which, owing to their very small eddy-current factors, tend to have large values of p. For ordinary dust cores, p varies roughly between 0·5 and 2·0.

47. Laminated coils with eddy-current shielding

Fig. 41 shows the type of Q_0 frequency curve obtained with a coil in which core eddy-current shielding is occurring. Curve (a) represents, as an example, the Q_0 of a gapped laminated core for which $\frac{\mu_0}{\mu_{0_g}} = 10$. The curve is plotted against the quantity θ, which serves as an index denoting the effect of eddy-current shielding. As θ is reduced, Fig. 19 suggests that Q_0 should continue to rise, but, in fact, a frequency is soon reached at which the d.c. resistance begins to have a controlling effect on Q_0, and the Q_0 frequency curve starts to fall again. At high frequencies the core losses predominate, and Q_0 tends to unity. Curve (*b*) represents another example in which the d.c. resistance factor is considerably greater than it was in the first one. In a case like this, Q_0 will never rise very much above 1. Inductance coils are very rarely used with laminated cores working under these conditions, which represent frequencies at which far better results could be obtained with dust-cores. The theory is mainly of interest in connection with the design of communication transformers which have to cover a wide band of frequencies.

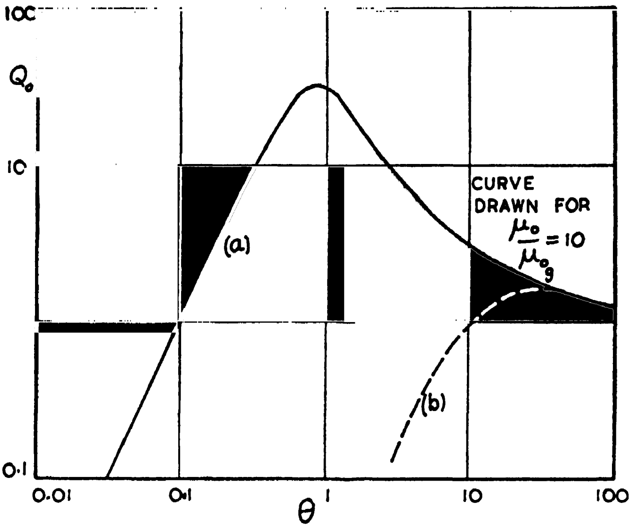

FIG. 41. Curves referred to in §47.

CHAPTER X

GRAPHICAL AIDS TO COIL DESIGN

48. "Equi-Q" charts

These charts, which, as their name suggests, are sets of curves of constant Q, plotted against frequency and inductance, were developed with the object of providing a rapid method of selecting, from a limited number of stock types, the best wire, core, number of turns, etc., to satisfy a given coil specification. Their preparation involves very little calculation, but they provide a method of presenting the results of impedance measurements, on a few sample coils, in such a way that it is possible to predict, by interpolation, the probable performance of any proposed coil, within the range of types covered by the charts.

49. Preparation of chart

Let us consider, for example, the construction of the "equi-Q" curves for coils wound on a certain pot-type dust core. To start with, the bobbin is wound with the maximum possible number of turns of a suitable wire, a note being made of the number of turns. Impedance measurements are then made at a sufficient number of frequencies to enable the Q frequency curve to be plotted over the desired working frequency range. Turns are then removed from the coil until the number of turns has been reduced to 75% of the maximum, and the process is repeated, giving another Q frequency curve.

Finally, by repeating again at 50% and 25% of the maximum turns, a set of curves, roughly of the form shown in Fig. 42, will be obtained. This diagram also shows, as an example, the way in which the $Q = 200$ equi-Q curve is projected from the Q frequency curves. The equi-Q curves should be smooth and free from sudden irregularities, a fact which provides a useful check on the measurements, since any serious error will displace the point concerned, so that it will be difficult or impossible to draw a smooth curve through it. Fig. 43 shows a completed equi-Q chart. The dotted lines have been added to show the amount by which the apparent series inductance has risen as a result of self-capacitance. The point A, for example, on the dotted curve, corresponding to 80% of the maximum turns, is plotted so that $\frac{AB}{BC}=\left[1-\left(\frac{f}{f_r}\right)^2\right]^{-1}$ (see § 8(*b*)). The main object of these dotted lines is to serve as a warning that the part of the diagram where their curvature is apparent must be avoided if coils are required whose series inductance is to be independent of frequency (e.g., coils for wave filters). In the chart which has been chosen as an example, the equi-Q lines tend to form complete loops, and show that the maximum possible Q is obtained, with this type of core, when the bobbin is wound with only about 80% of the maximum turns. This drop in Q as the remaining winding space is filled up is fairly typical of pot-type dust-cores, and becomes more noticeable as the working frequency is increased. It is caused by the capacitance between the

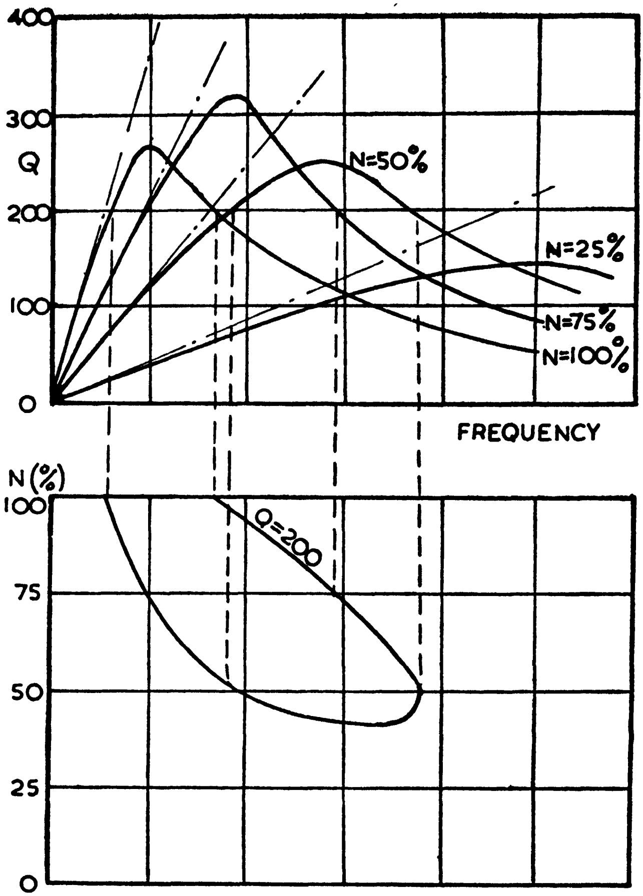

FIG. 42. Preparation of equi-Q charts.

outer layers of the winding and the inner surface of the dust-core shell which surrounds it. This added capacitance not only causes a direct reduction of Q, but also increases the core losses as a result of circulating currents which pass *via* this capacitance and the interparticle capacitances of the core itself.

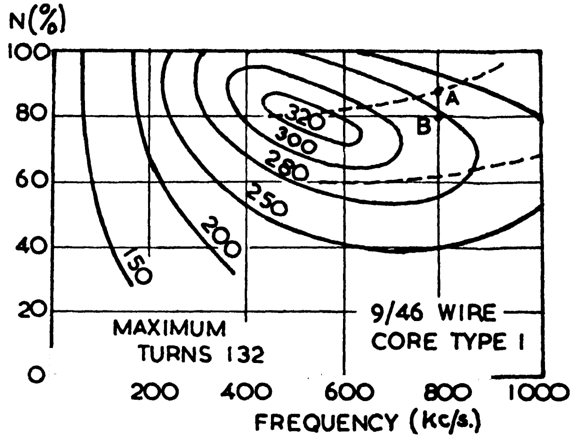

FIG. 43. A completed equi-Q chart.

50. Use of charts

Suppose that designs are required for the coils of a bandpass filter with a mid-band frequency of 500 kc./sec.; the Q values of all coils to exceed 250. It will be assumed that equi-Q charts have previously been prepared for several different stranded wires on each of, say, half-a-dozen standard types of dust core. A glance through these will then show that, in this case, the core corresponding to Fig. 43 is a suitable one, and that this particular wire will cover inductance values between about 0·3 and 1·0 mH, the approximate number of turns required for each inductance value being also given by the chart. In the same way, the best wire can then be selected for other inductances. It is not necessary for charts to be constructed for every possible wire type, a few representative samples generally being sufficient. Interpolation between these curves will then enable the results for intermediate wire types to be predicted with sufficient accuracy for most purposes.

It must be emphasized that the equi-Q chart method is not intended as a substitute for a detailed analysis, and the latter must still be used when designing from first principles. On the other hand, where there is no time for lengthy calculations, or where coils are required with a performance exceeding a certain minimum rather than with the best possible design, this method of predicting the probable behaviour of a proposed coil from the results of a limited number of measurements can be of great assistance.

51. Use of generalized Q_0 frequency curves

Suppose that the core constants of a coil are kept constant and the winding details are changed so that the space factor λ and the strand diameter d are variables. Assuming that the winding eddy-current loss is not necessarily negligible, this means that K and k will be variables, and therefore so will K and A in (213). B, on the other hand, will be fixed. Now, from (227),

$$\hat{Q}_0 = \hat{Q}_0' \left(\frac{2}{2+p}\right)$$

$$= \frac{\pi}{\sqrt{AK}}\left(\frac{2}{2+p}\right) \qquad . \quad . \quad . \quad . \quad . \quad . \quad (230)$$

Let us denote by $\hat{Q}_{0_1}$ the particular value of $\hat{Q}_0$ when $p=1$.

$$\text{i.e., } \hat{Q}_{0_1} = \frac{2\pi}{3\sqrt{AK}} = \frac{2\pi}{3B}$$

$$\text{or } \frac{\pi}{\sqrt{AK}} = \frac{3\hat{Q}_{01}}{2} \qquad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (231)$$

and, from (230),

$$\hat{Q}_0 = \hat{Q}_{0_1}\left(\frac{3}{2+p}\right)$$

$$\text{Also } f_0 = \sqrt{\frac{K}{A}} = \frac{K}{\sqrt{AK}}$$

$$= \frac{K}{B} \cdot \frac{B}{\sqrt{AK}} = \frac{K}{B} \cdot p \qquad . \quad . \quad . \quad . \quad . \quad (232)$$

so that, if f_{0_1} is the value of f_0 when $p=1$, we have

$$f_{0_1} = \frac{K}{B}$$

$$\text{or } f_0 = f_{0_1} \cdot p \qquad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (233)$$

We have now defined the co-ordinates of the peak of any Q_0 frequency curve in terms of the co-ordinates of the particular curve for which $p=1$, thus enabling a family of curves to be drawn, representing the effect on the Q_0 frequency curve as the winding specification of an iron-cored coil is changed, i.e., as A and K are changed (see Fig. 44). The whole series can be located relative to the Q_0 and frequency axes by calculating the co-ordinates of the reference point Y from

$$\hat{Q}_{0_1}=\frac{2\pi}{3B} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (234)$$

$$f_{0_1}=\frac{K}{B} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (235)$$

As A and K are varied, the appropriate curve is then picked out by calculating p from

$$p=\frac{B}{\sqrt{AK}}$$

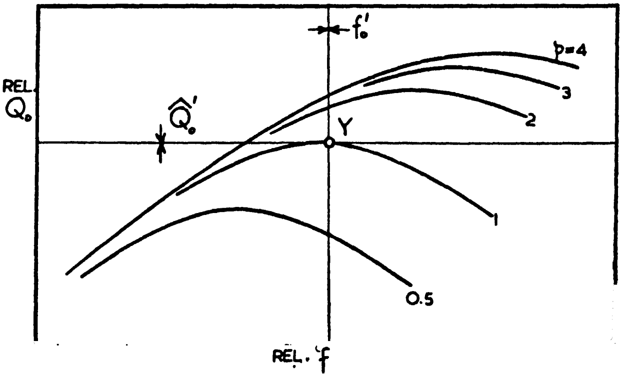

FIG. 44. Series of Q-frequency curves for an iron-cored coil (eddy-current shielding neglected).

This leads to another graphical method of design, not quite so general in its application as the equi-Q chart method, but yet often of assistance where, for example, a number of filter coils are all to be wound on the same type of core but all have different numbers of turns. Under these conditions, the above assumptions of constant B and changing A and K will be justified, and a general set of curves, of the type shown in Fig. 44, can be used. Fig. 45 shows in more detail a completed chart, prepared for a certain dust-core and indicating the method of selecting the correct curve and the frequency scale to be used with it. The intersections of the diagonal lines, representing frequencies, with the horizontal "K" lines, give the frequency calibration for each K value. (The diagonal lines are inclined at 45° and the " K " lines are spaced logarithmically to the same scale as that used for the frequencies, thus fulfilling the condition that f_{0_1} should be directly proportional to K.) Next, suppose that K is kept constant and the strand size is varied. The " p " values for various gauges can be calculated, since λ is fixed and k is proportional to d^2. These p values are then interpolated on the reference scale AB (Fig. 45) and projected horizontally on to a vertical extension of the appropriate " K " line (the " K = 1000 " line in the example chosen), thus

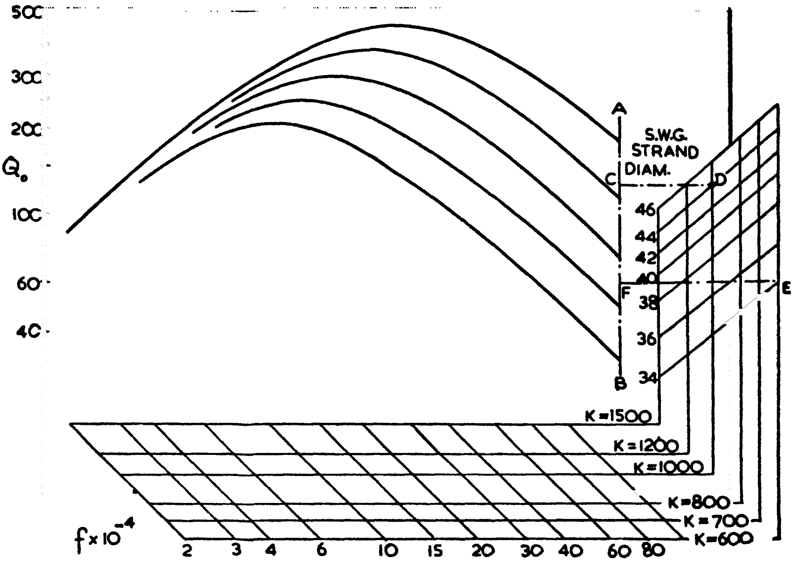

FIG. 45. Application of Fig. 44 to a particular core, taking into account variations in wire diameter and in the space factor of the winding.

giving the point D. A series of such points can be obtained in the same way and marked with the appropriate strand sizes. The whole procedure is then repeated with different values of K, and a new series of points such as D plotted on the corresponding " K " lines. Finally, as shown in the diagram, the plotted points can be joined by continuous lines.

To illustrate the use of a chart of this type, suppose that the Q_0 frequency curve is required for a winding, on the core under consideration, having strands of 34 S.W.G. wire and a space factor such that K = 600. The intersection of the " 34 S.W.G." line with the vertical " K = 600 " line gives the point E, which is projected across until it cuts the scale AB at F. The required curve is then known to pass through F, and can be interpolated quite simply. At the same time, the intersection of the "K = 600" line with the diagonal frequency lines gives the frequency scale to be used in conjunction with the above curve.

(*a*) GENERAL CURVES FOR AIR-CORED COILS

As p is made smaller and smaller, the diagonal S.W.G. lines in Fig. 45 become more and more nearly horizontal. In the limit, when $p = 0$, they are exactly horizontal, so that a definite curve can be allocated to each gauge size, irrespective of the value of K. Furthermore, all the curves are of the same shape, and can be plotted by calculating their peak co-ordinates for some chosen value of K. As K is changed, the appropriate change of frequency scale can be carried out as for the case of an iron-cored coil. An example of curves for an air-cored coil is shown in Fig. 46.

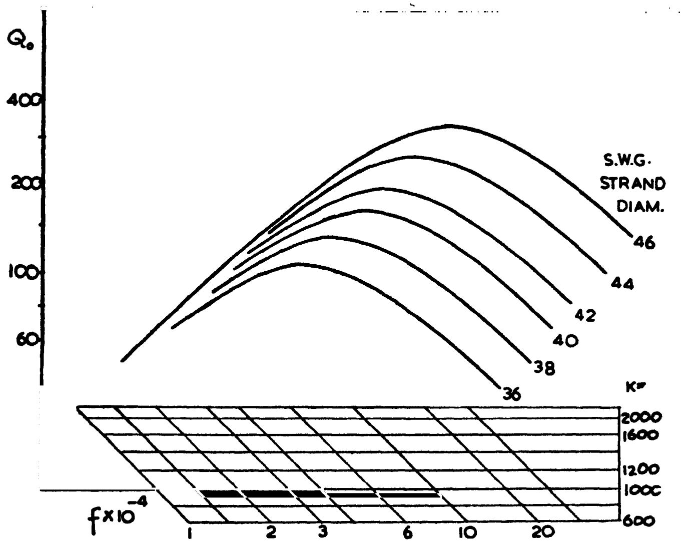

FIG. 46. Simplified form of Fig. 45 for air-cored coils.

(*b*) PLOTTING DEVICE FOR Q_0 FREQUENCY CURVES

Plate VII shows a device which can be used to apply the method described above. It consists essentially of vertical and horizontal cursors which slide along suitable guides, attached to a board on which a chart such as that shown in Fig. 45 is fixed. The operation requires little comment, although it will be noted that, as shown, one cursor has two vertical faces, the distance between which can be pre-set to correspond to a given frequency ratio. The original object of this was to enable the predicted Q_0 to be read off simultaneously for two different frequencies, one of which was that at which the coil was to work, and the latter a higher testing frequency, chosen for this purpose because at relatively high frequencies the eddy-current losses predominate and core or winding faults are more likely to show up.

CHAPTER XI

HARMONIC DISTORTION IN IRON-CORED COILS

The impedance of a coil with a ferromagnetic core is a function of the current flowing in the winding and, as a result, the waveform of an alternating voltage across the coil is never exactly the same as that of the corresponding alternating current. If the current is sinusoidal, the distorted voltage waveform can be analysed into

(i) a fundamental sinusoidal component in phase with the current
(ii) a similar component which is 90° out of phase with the current
(iii) a series of "harmonic" sinusoidal components whose frequencies are multiples of the fundamental.

Conversely, if a sinusoidal voltage is maintained across the coil, the current wave-form will be distorted and will contain harmonic components. This effect is often very troublesome, particularly in telecommunication equipment, since the passage of signal currents through an iron-cored coil will produce voltages at frequencies which did not exist in the original signal. With a single communication channel (e.g., a domestic radio receiver), the distortion will show up merely as a reduction in the quality of the received signal, but where several channels are transmitted simultaneously through common equipment (e.g., a carrier telephone system), harmonic distortion may cause interference between the various channels.

52. Distortion at low flux densities

Consider, first of all, a core which has no d.c. polarizing current in its winding, and which has been completely demagnetized by applying a large alternating magnetizing force, whose amplitude has then been reduced gradually to zero. At low flux densities, the hysteresis loop will be given by the Rayleigh loop equation

$$B = (\mu_0 + a\hat{H})\,H \pm \frac{a}{2}(\hat{H}^2 - H^2) \qquad . \quad . \quad . \quad (236)$$

If the current is sinusoidal, the instantaneous values of the magnetizing force are given by

$$H = \hat{H}\cos\omega t \qquad . \quad . \quad . \quad . \quad . \quad . \quad (237)$$

substituting for H in (236) gives

$$B=(\mu_0\hat{H}+a\hat{H}^2)\cos\omega t\pm\frac{a\hat{H}^2}{2}(1-\cos^2\omega t)$$

$$=(\mu_0\hat{H}+a\hat{H}^2)\cos\omega t\pm\frac{a\hat{H}^2}{2}\sin^2\omega t \quad . \quad . \quad (238)$$

The first term of this expression represents a sine wave, in phase with the H wave. The second term represents a wave whose positive half-cycles are proportional to $+\sin^2\omega t$ and whose negative half-cycles are proportional to $-\sin^2\omega t$. It is thus proportional to a function of ωt, such that

$$\begin{cases} f(\omega t)=\sin^2 wt \ (0\leq\omega t\leq\pi) \\ f(\omega t)=-\sin^2\omega t \ (\pi\leq\omega t\leq 2\pi) \end{cases} \quad . \quad . \quad . \quad (239)$$

This periodic function can be expanded as a Fourier series of the form

$$f(\omega t)=\frac{8}{\pi}\Sigma_1^\infty\frac{\sin(2n-1)\,\omega t}{(2n-1)\,[4-(2n-1)^2]} \quad . \quad . \quad (240)$$

where n is any positive integer.

Substituting of $n=1$, 2, 3, etc., shows that the first few terms of the series are

$$f(\omega t)=\frac{8}{3\pi}\sin\omega t-\frac{8}{15\pi}\sin 3\omega t-\frac{8}{105}\sin 5\omega t-\ldots (241)$$

The complete expression for the flux density at any instant then becomes

$$\begin{aligned} B &= (\mu_0\hat{H}+a\hat{H}^2)\cos\omega t \\ &+\frac{4a\hat{H}^2}{3\pi}\sin\omega t \\ &-\frac{4a\hat{H}^2}{3\pi}\left(\frac{1}{5}\sin 3\omega t+\frac{1}{35}\sin 5\omega t+\ldots.\right) \quad . \quad (242) \end{aligned}$$

The second term of (238) is thus seen to be made up, as expected, of a fundamental component 90° out of phase with the H wave, together with a series of odd-order harmonic waves (see § 19 (*d*)). The induced e.m.f. at any instant is proportional to the rate of change of flux density and is given by

$$e=NA\frac{dB}{dt}\,.\,10^{-8}\text{ volts} \quad . \quad . \quad . \quad . \quad . \quad (243)$$

It is thus necessary to differentiate (242) with respect to t.

$$\text{Then } f'(\omega t) = \frac{8\omega}{\pi} \Sigma_1^\infty \frac{\sin(2n-1)\,\omega t}{[4-(2n-1)^2]} \quad . \quad . \quad . \quad . \quad (244)$$

$$\text{and } \frac{dB}{dt} = (\mu_0 H + a\hat{H}^2)\,\omega \cos\left(\omega t + \frac{\pi}{2}\right)$$

$$+ \frac{4a\hat{H}^2\omega}{3\pi} \cos \omega t$$

$$- \frac{4a\hat{H}^2\omega}{3\pi}\left(\frac{3}{5}\cos 3\omega t + \frac{1}{7}\cos 5\omega t + \ldots .\right) \quad . \quad (245)$$

The instantaneous induced e.m.f. then becomes

$$e = (\mu_0 \hat{H} + aH^2)\,\omega \cos\left(\omega t + \frac{\pi}{2}\right) \text{NA} \,.\, 10^{-8}$$

$$+ \frac{4a\hat{H}^2\omega}{3\pi} \cos \omega t \,.\, \text{NA} \,.\, 10^{-8}$$

$$- \frac{4a\hat{H}^2\omega}{3\pi}\left(\frac{3}{5}\cos 3\omega t + \frac{1}{7}\cos \omega t + \ldots .\right) \text{NA} \,.\, 10^{-8} \quad (246)$$

The first term here is the component of the induced voltage, which is in quadrature with the current, and which therefore determines the inductance of the coil. If the inductance is L, the induced voltage is

$$L\frac{dI}{dt} = \omega\, L\hat{I} \cos\left(\omega t + \frac{\pi}{2}\right) \quad . \quad . \quad . \quad . \quad . \quad (247)$$

Comparing terms, it follows that

$$L\hat{I} = (\mu_0 \hat{H} + a\hat{H}^2)\,\text{NA} \,.\, 10^{-8}$$

$$\text{or } L = (\mu_0 + a\hat{H})\left(\frac{\hat{H}}{N\hat{I}}\right) N^2 A \,.\, 10^{-8} \quad . \quad . \quad . \quad (248)$$

$$\text{but } \frac{\hat{H}}{N\hat{I}} = \frac{4\pi}{10l}$$

$$\therefore L = (\mu_0 + a\hat{H}) \,.\, \frac{4\pi N^2 A}{l} \,.\, 10^{-9} \text{ henries}$$

$$= L_0\left(1 + \frac{a\hat{H}}{\mu_0}\right)$$

$$\text{or } \frac{L - L_0}{L_0} = \frac{a\hat{H}}{\mu_0} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (249)$$

The second term of (246) represents the power loss due to hysteresis. If the equivalent series resistance is R_h, the voltage drop across it will be

$$R_h\hat{I}\cos\omega t \qquad . \quad . \quad . \quad . \quad . \quad (250)$$

Comparing terms,

$$R_h = \frac{4\alpha\hat{H}^2\omega AN}{3\pi\hat{I}} \cdot 10^{-8} \qquad . \quad . \quad . \quad . \quad . \quad (251)$$

which reduces to

$$\frac{8\alpha\hat{H}fL_0}{3\mu_0} \qquad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (252)$$

so that, from (249),

$$\frac{R_h}{L - L_0} = \frac{8}{3}f \qquad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (253)$$

The series representing the harmonic components can be expressed in terms of the hysteresis resistance R_h by rewriting (246) in the form

$$\begin{aligned} e &= \omega L\hat{I}\cos\left(\omega t + \frac{\pi}{2}\right) \\ &\quad + R_h\hat{I}\cos\omega t \\ &\quad - R_h\hat{I}\left(\frac{3}{5}\cos 3\omega t + \frac{1}{7}\cos 5\omega t + \ldots\right) \qquad . \quad . \quad (254) \end{aligned}$$

or as

$$\begin{aligned} e &= \omega L\hat{I}\cos\left(\omega t + \frac{\pi}{2}\right) \\ &\quad + R_h\hat{I}\cos\omega t \\ &\quad - [\hat{E}_3\cos 3\omega t + \hat{E}_5\cos 5\omega t + \ldots . \\ &\quad \hat{E}_{(2n-1)}\cos(2n-1)\,\omega t] \qquad . \quad . \quad . \quad (255) \end{aligned}$$

where

$$E_3 = \frac{3}{5}R_h\hat{I}$$

$$E_5 = \frac{1}{7}R_h\hat{I}$$

$$\hat{E}_{(2n-1)} = \frac{3R_hI}{[4 - (2n-1)^2]} \qquad . \quad . \quad . \quad . \quad . \quad (256)$$

Finally, since

$$R_h = F_hIfL_0^{\frac{1}{2}} \qquad . \quad . \quad . \quad . \quad . \quad . \quad (257)$$

The r.m.s. value of the $(2n-1)$th harmonic e.m.f. is

$$E_{(2n-1)} = \frac{3F_hI^2fL_0^{\frac{1}{2}}}{[4 - (2n-1)^2]} \qquad . \quad . \quad . \quad . \quad . \quad (258)$$

It is interesting to note that the amplitudes of all these odd-order harmonics are proportional to the hysteresis factor, and also to the *square* of the fundamental current.

(*a*) DISTORTION DUE TO A SHUNT COIL

Suppose that a generator of e.m.f. E_1 and internal impedance R, feeds a matched load impedance R, while an iron-cored coil of inductance L is shunted across the load. To make the problem simpler, it will be assumed that the impedances are non-reactive resistances and the inductance is independent of frequency. If ωL is large compared with R, the voltage across the coil will be sinusoidal and equal to $\frac{E_1}{2}$, while the current flowing in it will be distorted. The fundamental component of the current will be

$$I_1 = \frac{E_1}{2\omega L} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (259)$$

This will produce a series of harmonic e.m.f.s, of which the third, denoted by E_3, will be taken as an example.

$$\text{Then } E_3 = 0{\cdot}6\ F_h I^2 L_0 f$$

$$= \frac{0{\cdot}3}{4\pi} \cdot \frac{F_h E_1{}^2}{\omega L^{\frac{1}{2}}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (260)$$

This is the harmonic e.m.f. which is produced by the fundamental component of the current, acting alone. Theoretically, it should be necessary to make a correction to this result, to take into account the additional e.m.f. at the third harmonic frequency, produced by the harmonic components of the distorted current wave, but in practice, it is found that the accuracy with which most materials conform to the Rayleigh hysteresis theory is not sufficient to make this additional complication worth while. The current at the third harmonic frequency will divide equally between the load and the generator impedances, so that the third harmonic voltage across the load will be

$$V_3 = E_3 \cdot \frac{R}{6\omega L} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (261)$$

The fundamental voltage across the load is

$$V_1 = \frac{E_1}{2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (262)$$

$$\text{Now } V_3 = \frac{0{\cdot}3}{24\pi} \cdot \frac{F_h E_1{}^2 R}{\omega^2 L^{\frac{1}{2}}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (263)$$

$$= \frac{0{\cdot}3}{6\pi} \cdot \frac{F_h V_1{}^2 R}{\omega^2 L^{\frac{1}{2}}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (264)$$

so that the " signal/noise " ratio is

$$\frac{V_3}{V_1}=\frac{0{\cdot}3}{6\pi}\cdot\frac{F_hV_1R}{\omega^2L^{\frac{1}{2}}} \quad . \quad . \quad . \quad . \quad . \quad (265)$$

From this it follows that, if R, V_1 and ω are constant, $\frac{V_3}{V_1}$ is proportional to $\frac{F_h}{L^{\frac{1}{2}}}$. Thus, if the coil has an air-gap, $\frac{V_3}{V_1}$ is proportional to $\frac{F_{h_g}}{L_g^{\frac{1}{2}}}$ which, as shown by (133), is independent of the air-gap ratio. This result has an important bearing, for example, on the design of anode coupling chokes in valve amplifiers, since it shows that, once the maximum number of turns has been fixed by the maximum allowable d.c. resistance of the choke winding, no change in harmonic distortion will be caused by varying the air-gap. The latter can thus be adjusted to give the maximum inductance for a given number of turns, with the appropriate d.c. anode current flowing through the winding (see § 23 (*b*)). Once the optimum conditions have been reached, the only ways in which the harmonic distortion can be reduced are to increase the core size, to increase the number of turns by reducing the valve anode current, or to allow a larger d.c. voltage drop across the choke. It is also interesting to note that, for a given core size, the use of a push-pull amplifier circuit not only reduces even-order harmonics to a low level, but also, by removing the polarizing magnetizing force from the core, enables a higher inductance to be obtained for a given number of turns and thus improves the signal/noise ratio arising from odd-order harmonics. If, as in the case of a coil for a wave filter, the value of L is kept constant by changing the number of turns as the air-gap ratio is changed, $\frac{V_3}{V_1}$ is proportional to F_{h_g}, i.e., to $(1+\mu_0\beta)^{-\frac{1}{2}}$, showing how, for a given inductance value, the harmonic distortion is reduced by an air-gap.

(*b*) DISTORTION DUE TO A SERIES COIL

Consider now a coil, of inductance L, connected in series with the generator and load. ωL is assumed to be small compared with R.

$$\text{Now } I_1=\frac{E_1}{2R}$$

$$E_3=0{\cdot}6\,F_hI_1^2L^{\frac{1}{2}}f$$

$$\frac{0{\cdot}6}{2\pi}F_hI_1^2L^{\frac{1}{2}}\omega \quad .$$

The third harmonic current in the load will be

$$I_3=\frac{E_3}{2R}$$

$$=\frac{0{\cdot}6}{4\pi}\cdot\frac{F_h I_1{}^2 L^{\frac{3}{2}}\omega}{R}$$

so that the " signal/noise " ratio is

$$\frac{I_3}{I_1}=\frac{0{\cdot}6}{4\pi}\cdot\frac{F_h I_1{}^2 L^{\frac{3}{2}}\omega}{R}$$

If R, I_1 and ω are constant, $\frac{I_3}{I_1}$ is proportional to $F_h L^{\frac{3}{2}}$ so that for an air-gap $\frac{I_3}{I_1}$ is proportional to $(1+\mu_0\beta)^{-3}$.

For both shunt and series coils, if the inductance is constant, the harmonic distortion is proportional to the hysteresis factor. For cores with air-gaps, the latter is itself proportional to $\mu_g{}^{\frac{3}{2}}$ (volume)$^{-\frac{1}{2}}$. This leads to the important rule that, for a given core material and a fixed inductance, the odd-order harmonic distortion caused by hysteresis is proportional to $\mu_g{}^{\frac{3}{2}}$, and inversely proportional to the square root of the core volume.

53. Compromise between harmonic distortion and Q value

Suppose that the core volume of an iron-cored coil has been fixed by space considerations, cost, etc. Then, as shown in § 46, there will be an optimum gap ratio for which the Q at a given frequency will be a maximum. From the result obtained above, it is clear that, having fixed the inductance, the gap ratio and the core size, the distortion (for a given material) is also defined.

It follows therefore that if the harmonic distortion is outside the allowable limits, after the gap-ratio has been adjusted to give the maximum Q, it will be necessary to increase the gap ratio beyond the optimum. The resulting reduction in distortion is then obtained at the expense of the Q value of the coil.

54. Variation of impedance over each cycle of alternation

Consider once again Fig. 16, which was used in Chapter V to illustrate the formation of a hysteresis loop, and the way in which it causes waveform distortion. Let us suppose for a moment that the hysteresis loop of some imaginary material is an ellipse, with its major axis lying along the line P_1P_2 (Fig. 15). The " difference " wave, representing the effect of the loop, will now be sinusoidal. It passes through zero when H is a maximum, and reaches a maximum as H passes through zero, so that it is clearly 90° out of phase with the H wave. The corresponding induced e.m.f. will be in phase with the current, and represents the power dissipated as a result of the presence of the loop. In the special case of an elliptical loop, there will be no harmonic distortion produced. If the hysteresis loop has any other shape, the " difference " wave will no longer be exactly sinusoidal, but will become distorted. We have already discussed one way of interpreting this distortion ; that of defining the coil impedance in terms of linear inductance and resistance components, and then imagining the harmonic e.m.f.s to be due to small generators, of negligible internal impedance, in series with the winding. An alternative method is to regard the distortion as being due to a variation of the coil impedance throughout each cycle of alternation.

If the Rayleigh loop equations hold, the instantaneous induced voltage, when a sinusoidal current flows in the winding, is

$$\begin{aligned} e &= \mathrm{NA}\frac{d\mathrm{B}}{dt} \,.\, 10^{-8} \\ &= \mathrm{NA}\frac{d}{dt}\left[(\mu_0\hat{\mathrm{H}} + a\hat{\mathrm{H}}^2)\cos\omega t \pm \frac{a\hat{\mathrm{H}}^2}{2}(1-\cos^2\omega t)\right] \\ &= \mathrm{NA}\frac{d}{dt}\left[(\mu_0\hat{\mathrm{H}} + a\hat{\mathrm{H}}^2)\cos\omega t \pm \frac{a\hat{\mathrm{H}}^2}{2}\sin^2\omega t\right] \\ &= -\mathrm{NA}\,(\mu_0\hat{\mathrm{H}} + a\hat{\mathrm{H}}^2)\,\omega\sin\omega t \pm a\hat{\mathrm{H}}^2\mathrm{NA}\,\omega\sin\omega t\cos\omega t) \qquad (266) \end{aligned}$$

The second term corresponds to a varying resistance, whose instantaneous value is r, where

$$\begin{aligned} r = \frac{e}{i} &= \pm\frac{a\hat{\mathrm{H}}\mathrm{NA}\,\omega\sin\omega t\cos\omega t}{\hat{\mathrm{I}}\cos\omega t} \\ &= \pm\frac{a\hat{\mathrm{H}}\mathrm{NA}}{\hat{\mathrm{I}}}\,\omega\sin\omega t \qquad . \quad . \quad . \quad . \quad (267) \end{aligned}$$

The resistance is proportional to $\pm\omega\sin\omega t$, and therefore numerically proportional to the voltage across an ideal inductance, the sign being chosen so that the resistance is always positive. We have thus derived another equivalent circuit for an iron-cored coil, working at low flux densities. This consists of a linear inductance, in series with two resistance components, one of which is linear and represents the d.c. resistance, eddy-current losses, etc., while the other has an instantaneous value which is always positive but is numerically equal to the instantaneous voltage across the inductance. It is worth noting that this varying resistance component is introduced to represent the waveform distortion ; it does not correspond to the change in effective resistance which takes place as the *amplitude* of the flux density is changed. The latter is caused by a change in the total power dissipated and depends on the change in area of the hysteresis loop. Looking at the matter from another point of view, we could state that the resistance component which varies over the cycle represents the deviation of the shape of the hysteresis loop from an ideal elliptical form, while the normal hysteresis resistance depends merely on the area of the loop, and not on its shape. For the Rayleigh loop, each half-cycle of the " difference " wave (see Fig. 16), representing the hysteresis distortion, is symmetrical about $\omega t = \frac{\pi}{2}, \frac{3\pi}{2}$, etc. As the flux density is increased outside the limiting value below which the Rayleigh approximation holds, the " difference " wave will begin to assume the form shown in the diagram, and this symmetry will disappear. As a result, the equation corresponding to (266) will contain a series of cosine terms on the right-hand side, in addition to the sine terms, so that the *inductance* will also appear to vary slightly over each cycle. In general, therefore, the impedance of an iron-cored coil can be represented by an equivalent circuit consisting of a series combination of a resistance and an inductance, both of which vary slightly over each cycle of alternation. When the flux density is low enough for the Rayleigh equations to hold, the variation in the inductance component becomes negligible.

55. Distortion at high flux densities and high frequencies

When the frequency is high enough for eddy-current shielding to occur, the flux density near the surface of each lamination may be many times the average for the whole lamination. If the magnetizing force is very small, the flux density over the whole cross-section may be still within the Rayleigh region. Under these conditions, it is possible to estimate the harmonic e.m.f. by imagining the laminations to be split up into layers, over each of which the flux density can be assumed to be constant. In this way it can be shown that the effective value of F_h to be used in (258) begins to fall almost inversely with frequency for values of θ greater than 1. As the flux density is increased, at frequencies such that eddy-current shielding is taking place, a point will eventually be reached at which magnetic saturation is beginning to occur in the outer layers of each lamination and, owing to the concentration of flux, saturation will begin when the average flux density is still below the saturation level. As the flux density is still further increased, saturation will reduce the permeability of the outer layers to a relatively small value, thus tending to destroy their shielding effect and allowing the depth of penetration to increase. We thus expect to find that, at high average flux densities, the effect of eddy-current shielding will become progressively less.

56. Production of even harmonics

Even-order harmonics are produced only if there is a lack of symmetry between the positive and negative lobes of the hysteresis loop. In all normal ferromagnetic materials, when completely demagnetized, the hysteresis loop is sufficiently symmetrical to make all the even-order harmonics practically negligible. The application of a small polarizing magnetizing force, or even the existence of residual magnetism as a result of the past magnetic history of the core, will cause a great increase in the level of even-order distortion, but will have a relatively small effect on the odd-order distortion produced by the loop. Even-order distortion is thus difficult to predict, and may depend on previous treatment of the core, or on the presence of a steady polarizing magnetizing force (possibly even the Earth's field). It does not necessarily obey a simple square law, even at low flux densities.

CHAPTER XII

IMPEDANCE MEASUREMENTS

The impedance of an inductance coil at various frequencies can be measured by several different methods, the choice of which will depend to a great extent on the purpose for which the measurements are being made. In many cases, for example, it is the inductance of the coil which is of importance and which must be measured accurately, while a mere rough estimate of the effective series or shunt resistance is all that is required. On the other hand, if measurements are being made on an iron-cored test coil, with the object of determining the loss coefficients of the core material, the loss resistance will have to be measured much more accurately. The accuracy required would be greater for a dust-cored coil with small loss coefficients than it would be for an ungapped laminated-cored coil, whose losses would be greater and therefore easier to measure. Practically all methods of impedance measurement depend, directly or indirectly, on the use of some calibrated impedance standard, which is used as a reference. Exceptions to this rule might be the measurement of the modulus of a coil impedance by direct comparison of the voltage drop across its terminals and the current flowing through it, or the measurement of the losses in a coil by measuring the heat produced by a given current. Even in these cases, the calibration of the meters used would probably depend eventually on that of some impedance standard. Generally, therefore, it can be stated that the accuracy of any impedance measurement depends on

(i) the accuracy of calibration of the standard
(ii) the accuracy of the comparison between the impedance of the test coil and that of the standard.

Errors in the calibration of the standard are often the main source of inaccuracies in coil measurements.

57. The Q-meter

This instrument, as its name implies, gives a direct reading of the Q of a coil. The test coil is connected in series with a variable condenser which forms the impedance standard. A known alternating voltage, at the testing frequency, is applied across the series tuned circuit, and the latter is adjusted to resonance, as indicated by a maximum reading on a valve-voltmeter which is connected across the condenser. The impedance of the condenser will be very nearly a pure reactance which, at resonance, is numerically equal to that of the test coil. The ratio of the modulus of the voltage across the condenser to that of the e.m.f. applied to the circuit is thus $\frac{\omega L}{R+R_S}$, where R_S is the equivalent series loss resistance of the condenser. (The suffix $_S$ will be used throughout to denote quantities relating to the impedance standard.)

$$\text{Now } \frac{\omega L}{R+R_S}=\left(\frac{R}{\omega L}+\frac{R_S}{\omega L}\right)^{-1} \qquad (268)$$

$$\text{and } \frac{R_S}{\omega L}=R_S\omega C_S=\frac{L}{Q_S} \qquad (269)$$

$$\therefore \frac{\omega L}{R+R_S}=\left(\frac{1}{Q}+\frac{1}{Q_S}\right)^{-1}$$

$$=Q\left(1+\frac{Q}{Q_S}\right)^{-1} \qquad (270)$$

which, for small values of $\frac{Q}{Q_S}$, is very nearly equal to

$$Q\left(1-\frac{Q}{Q_S}\right) \qquad (271)$$

If the e.m.f. applied to the circuit is kept constant, the readings of the voltmeter, shunted across the condenser, will be proportional to $Q\left(1+\frac{Q}{Q_S}\right)^{-1}$ and can be calibrated to read directly in terms of this quantity. The instrument will give direct readings of the Q of the test coil, provided Q_S is much greater than Q. If $\frac{Q}{Q_S}$ cannot be neglected, the value of Q_S must be known, so that the necessary correction can be made to the readings. A numerical example will be given to illustrate this point, since it is one which arises in connection with all methods of impedance measurement which involve the use of coils or condensers as reactance standards. The loss in the reactance standard must either be negligible or accurately known.

Example

Suppose that a test coil, with an inductance of 10 millihenries, gives a Q-meter reading of 300 at a frequency of 100 kc./sec. The capacity of the tuning condenser at resonance will be 0·000253 μF, and therefore will be within the range of an air-dielectric variable condenser, whose Q should be at least 10,000. The error in the indicated value of Q would then be 3%. A coil of the same inductance, measured at 10 kc./sec., would require a tuning capacity of 0·0253 μF, most of which would probably be provided by a step-by-step solid-dielectric variable condenser, with an air-dielectric condenser as a trimmer. The Q of a mica condenser may be as low as 1,000, so that if this coil also gives a Q-meter reading of 300, the error will be more like 30%. In this case it would be necessary to determine the exact Q of the condenser, at this particular frequency and capacity setting, and to correct the reading accordingly. Alternatively, the use of the solid-dielectric condenser could be avoided by introducing, as an auxiliary reactance standard, another coil, which is so designed that its series resistance can be accurately calculated and allowed for. This is called a " partial substitution " method. One disadvantage of the Q-meter is that its accuracy depends on meter calibrations and on the characteristics of thermionic valves, rectifiers, etc. In spite of this, it is a very useful instrument for routine testing of coils, since it is simple to operate and gives a direct reading. If measurements are made at a single frequency only, the variable condenser can be calibrated directly in terms of coil inductance.

58. Impedance-variation method

In this method of impedance measurement, the test coil is connected in series with a calibrated condenser and the current in this resonant circuit is maintained at a constant value. A voltmeter is connected across the circuit and the reading is noted. One method of procedure is then to place, in series with the coil, a calibrated resistance which is adjusted until the voltage across the whole circuit, at resonance, is exactly twice what it was before. The loss resistance of coil and tuning condenser is then equal to that indicated by the calibrated resistance standard, while the inductance of the test coil can be calculated from the setting of the tuning condenser. An alternative method is to " detune " the resonant circuit by changing the condenser setting until the voltage across the circuit rises to some predetermined multiple of its original value. The total loss resistance of the coil and condenser can then be calculated from the change in capacity required. An important point to note here is that it is only the Q of the capacity *difference* which enters into calculation, so that it should be possible to arrange that the change of capacity is carried out by a readjustment of the air-dielectric trimmer condenser only. As pointed out above, the loss in a condenser of this type can usually be neglected, so that no further correction will then be necessary.

The accuracy of impedance-variation methods depends, not only on the calibration of one or two reactance standards, but also on the accuracy of a meter calibration.

59. Resistance-neutralization method

In this method, a variable condenser is shunted across the test coil and adjusted to the approximate value required for resonance at the test frequency. This parallel resonant circuit is then shunted across a two-terminal network, which contains a source of power (e.g., thermionic valves) and whose impedance is non-reactive but *negative*. The theory of negative impedance is outside the scope of this book, but the idea can perhaps be illustrated in the following way. Consider a simple valve oscillator, using a parallel tuned circuit. We know that, during oscillation, power is dissipated in the coil and condenser of the tuned circuit, and that, to maintain oscillation, the rest of the circuit must supply an equal amount of power. If the tuned circuit is removed, the impedance, looking back into the terminals to which it was connected, must be of a type which *supplies* power, and therefore absorbs a *negative* amount of power. An impedance of this type is called a negative impedance, and, when non-reactive, will be a negative resistance. When a voltage is applied across its terminals, it will cause a current to flow, which will be 180° out of phase with the voltage. The method of using such a device to measure the equivalent shunt resistance of the tuned circuit will now be clear. The value of the negative resistance (which is assumed to have been calibrated in some way) is adjusted by altering the value of one of the components of the negative resistance circuit until oscillation starts, and the tuning condenser is then set to give the desired frequency. Next, the negative resistance is increased until oscillation just stops (as it must do eventually, because an infinite negative resistance is the impedance of a device which will supply no current, and therefore no power, whatever voltage is applied to it). A decrease of negative resistance will cause the oscillation to start up again, and two settings can be found, one for which oscillation just starts, and one for which it just stops. The mean of these settings is then taken as numerically equal to the shunt resistance of the tuned circuit, and can be used to calculate the Q of the test coil, whose inductance is given by the condenser setting and the frequency. It has been claimed* that this method of coil measurement has advantages at very high frequencies, at which other methods are difficult to apply, owing to stray capacitances and the inductance of the connecting leads.

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* Inuma, in *Proc. I.R.E.*, March, 1940.

60. The Wheatstone bridge

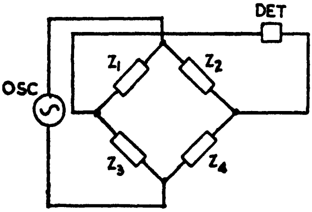


FIG. 47. The basic a.c. bridge circuit.

A number of methods of impedance measurement can be developed from the simple Wheatstone bridge circuit, the essentials of which are shown in Fig. 47. An alternating e.m.f. at the testing frequency is applied, between the points A and B, while a detector of some sort is used to indicate the difference of a.c. potential which appears between the points C and D. When a certain relationship exists between the impedances which form the bridge arms, the detector will give zero reading, showing that the points C and D are at the same potential. The bridge is then said to be balanced, the condition for this to occur being

$$Z_1Z_4=Z_2Z_3 \quad . \quad . \quad . \quad . \quad . \quad . \quad (272)$$

If the unknown impedance forms one of the bridge arms, its value can thus be calculated from a knowledge of the values of the other impedances when the bridge is balanced. For example, if Z_4 is the unknown :

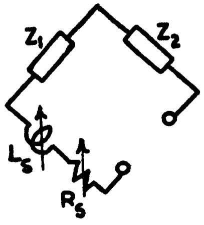
$$Z_4=Z_3\left(\frac{Z_2}{Z_1}\right)=\frac{1}{Z_1}\cdot(Z_2Z_3) \quad . \quad . \quad . \quad . \quad (273)$$

We could then use Z_3 as the impedance standard, the result depending only on the ratio of the impedances Z_1 and Z_2, which would be termed the " ratio-arms " of the bridge. Alternatively, if $\frac{1}{Z_1}$ were made the impedance standard, the result would depend on the product of the " product arms " Z_2 and Z_3.

(a) SENSITIVITY OF BRIDGE CIRCUITS

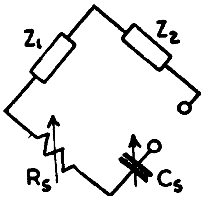
The sensitivity of adjustment of a bridge depends on the ratio of the voltage which appears at the detector terminals to the voltage applied by the generator, when the impedance of one bridge arm is changed by a given percentage from its balance value. It can be shown that this sensitivity will be a maximum when the input impedance of the detector is high compared with the impedance of the bridge arms, and also that the sensitivity is proportional to the modulus of the quantity $\dfrac{\frac{Z_1}{Z_3}}{\left(1+\frac{Z_1}{Z_2}\right)^2}$.

(*a*) *COMPARISON*



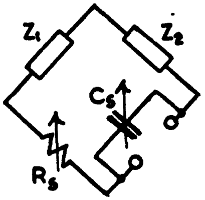
$$R = R_S\frac{Z_2}{Z_1}$$
$$L = L_S\frac{Z_2}{Z_1}$$

(*b*) *SERIES RESONANT*



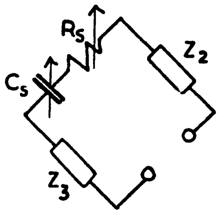
$$R = R_S\frac{Z_2}{Z_1}$$
$$L = \frac{1}{\omega^2 C_S}\,\frac{Z_2}{Z_1}$$

(*c*) *PARALLEL RESONANT*



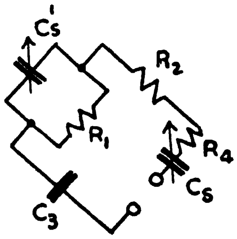
$$R' = R_S\frac{Z_2}{Z_1}$$
$$L' = \frac{1}{\omega^2 C_S}\,\frac{Z_2}{Z_1}$$

(*d*) *HAY*



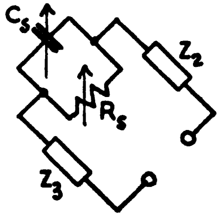
$$R' = \frac{Z_2 Z_3}{R_S}$$
$$L' = Z_2 Z_3 C_S$$

(*f*) *SCHERING*



$$R = \frac{R_2}{C_3}\left(\text{Change of } C_S'\right)$$
$$L = \frac{1}{\omega^2\,(\text{Change of } C_S)}$$

(*e*) *MAXWELL*



$$R = \frac{Z_2 Z_3}{R_S}$$
$$L = Z_2 Z_3 C_S$$

FIG. 48. Some applications of the a.c. bridge principle, showing the balance conditions in each case.

The following conditions may apply:

(i) Z_1 and Z_3 are reactances of the same sign
(ii) Z_1 and Z_3 are reactances of the opposite sign
(iii) Z_1 is reactive and Z_3 is resistive (or *vice-versa*).

In each case the sensitivity will be a maximum when the modulus of the ratio of Z_1 and Z_3 is unity. Stated in words, this means that, for the best results to be obtained, the modulus of the impedance of the arm which is placed, in series with the unknown, across the generator, should have approximately the same value as that of the unknown impedance.

To obtain the maximum voltage across the points A and B for a given generator power, the internal impedance of the generator should be matched to the impedance presented to it by the bridge network when balanced.

(*b*) BRIDGE CIRCUITS

The various types of simple a.c. bridge differ only in the types of impedance standards used, and in the relative positions in the bridge arms of the standards and the unknown impedance. Some possible bridge arrangements are shown in Fig. 48, which also gives the balance conditions in each case. The resonant bridges (*b*), (*c*) and (*f*), differ from the others in that their balance depends critically on the testing frequency, which has to be controlled to a greater degree of accuracy than is necessary for the non-resonant types. Bridge circuit (*f*) is interesting, because it enables both the resistive and inductive components of the unknown impedance to be measured in terms of *differences* in the settings of the variable condensers and while all the resistances are fixed. This is an advantage at high frequencies, where an air-dielectric condenser forms a more satisfactory impedance standard than a variable resistor. This is a substitution bridge, in which an initial balance is made with the " unknown " terminals short-circuited, and the bridge then balanced again with the unknown impedance inserted.

The choice of the best bridge circuit to be used for a given purpose depends on the frequency range and the range of inductance and Q values which are to be measured. The bridge must be arranged to give the necessary sensitivity, and the fixed impedances chosen, so that the values required for the standard resistance and reactance fall within the range of available components. The question of electrostatic screening of the bridge components is of great importance at higher frequencies, and may have a deciding influence on the choice of method. Generally it can be stated that the resonant bridges are most suitable for the accurate measurement of the effective resistance of inductance coils, while the non-resonant types are useful for accurate inductance measurements. The accuracy of any of these bridge types can be improved by the use of a substitution method, in which the bridge is initially balanced and the *change* in the settings of the standard impedances noted, as the unknown impedance is inserted into the circuit. In this way, the effect of stray capacitances and the resistance and inductance of the connecting leads is reduced to a minimum.

(c) OSCILLATORS AND DETECTORS

The requirements relating to oscillators for use with impedance bridges are most stringent in the case of resonant bridges. This is partly because the balance of such bridges depends critically on the testing frequency, thus necessitating a high order of frequency stability, and partly because the bridge is balanced to a single frequency, and completely unbalanced to harmonics or noise in the output of the oscillator. Experience has shown that some mains-driven oscillators, although perfectly satisfactory for other purposes, produce a hum which is noticeable only when used with a resonant bridge near balance and which cannot be filtered out. This effect has been shown to be the result of frequency modulation of the oscillator output by the mains frequency. It can be cleared up by changing to a battery-driven oscillator, or by paying careful attention to the smoothing of the h.t. supply.

Detectors may be arranged to give either visible or audible indications of the balance condition. The choice is largely a matter of personal preference, and it may be thought desirable to combine the two methods, using perhaps a " magic-eye " indicator for rough balance adjustments, and sensitive headphones for the final adjustment. The detector will usually contain a frequency-changer circuit for transforming the input signal to a convenient audio range (about 800 to 1000 c./c.). It is essential to provide some sort of frequency discrimination between the bridge and the detector input, particularly when resonant bridge types are used. This is because errors will be caused if harmonic voltages are allowed to pass into the frequency-changer, where, by intermodulation between themselves, they can cause a spurious " fundamental " signal. This signal will be heard even when the bridge is correctly balanced, and its presence will cause the operator to continue to adjust the bridge in an attempt to balance it out. This readjustment will alter the relative phases of the harmonics, and thus change the level of the unwanted fundamental which is being produced in the frequency-changer. The final result is a condition in which it may be impossible to find a balance, although there may be more than one point at which the output drops to a minimum, indicating an apparent balance. Symptoms of this sort can be taken as an indication that better discrimination against harmonics is required at the input to the detector. It should be noted that reduction of the harmonic level in the oscillator output may not clear up the trouble, because, if iron-cored coils are being tested, the latter may themselves cause enough harmonic distortion to upset the results. When headphones are used, some sort of voltage limiting circuit should be included to prevent painful acoustic shock if the bridge is accidentally unbalanced when working with full sensitivity.

61. " Twin-T " circuits

Fig. 49 (*a*) shows the fundamental arrangement of another type of circuit* which can be used for impedance measurements. As in the case of the simple Wheatstone bridge, there is a balance condition, for which no transfer of energy can take place between the input and output of the circuit. The name " twin-T " is derived from the fact that the circuit is made up of two " T " networks connected in parallel. If the output terminals are closed by any impedance, and a generator connected to the input terminals, the current in the load will be the sum of the currents which each component bridge would have produced alone. The balance condition, therefore, simply means that these two currents are arranged to have the same amplitude but

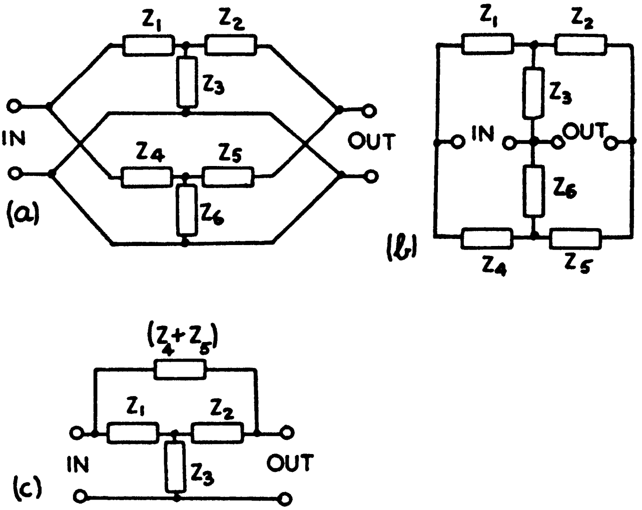


FIG. 49.

(*a*) The basic "twin-T" circuit.
(*b*) Application to impedance measurements.
(*c*) The "bridged-T" circuit.

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* Sinclair, in *Proc. I.R.E.*, July, 1940.

to be 180° out of phase with each other, so that they cancel out. The condition for this to happen is

$$\left(Z_1+Z_2+\frac{Z_1Z_2}{Z_3}\right)=-\left(Z_4+Z_5+\frac{Z_4Z_5}{Z_6}\right) \quad . \quad . \quad . \quad (274)$$

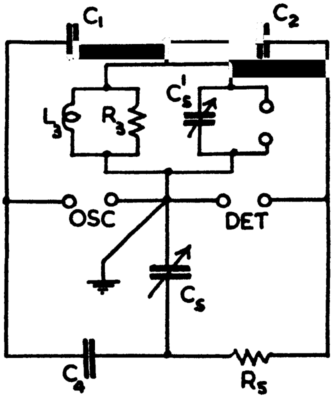
If the impedance Z_6 is made infinite, the circuit configuration becomes that shown in Fig. 49 (*c*), this arrangement being called a " bridged-T " circuit. The conditions for balance then reduce to

$$\left(Z_1+Z_2+\frac{Z_1Z_2}{Z_3}\right)=-(Z_4+Z_5) \quad . \quad . \quad . \quad . \quad (275)$$

Application to Impedance Measurements

Circuits of the type described can be used for the measurement of an unknown impedance by adjusting the circuit to balance initially and then altering the component values to attain a new balance when the unknown impedance has been inserted. The main advantage of the " Twin-T " circuit over the simple bridge is that the input, the output, and two of the impedances, all share a common terminal, which can be connected to earth (see Fig. 49 (*b*)). This simplifies the problem of electrostatic screening.

An example of a " Twin-T " circuit for coil measurements is shown in Fig. 50. L_3 is a fixed coil, whose resistance is included in the resistance R_3. The only variable components are the condensers C_S and C_S', which form the impedance standards. These condensers are



$$\frac{1}{R^1}=\left(\frac{R_5\omega^2C_1C_2}{C_4}\right)\left(\text{Change in } C_S\right)$$

$$\frac{1}{L'}=\omega^2\left(\text{Change in } C'_S\right)$$

Fig. 50. Details of the "twin-T" circuit, as applied to coil measurement. The formulae indicate the method of calculating the coil impedance.

first of all adjusted to obtain a balance, as indicated by a zero reading of the detector, at the test frequency but with the " unknown " impedance disconnected. The latter is then connected in parallel with C_S' and a new balance is obtained. If the unknown is represented by its shunt resistance R′, and shunt inductance L′, the value of R′ and L′ can be calculated from the equations

$$\frac{1}{R'}=\left(\frac{R_S\omega^2C_1C_2}{C_4}\right)\delta C_S \quad . \quad . \quad . \quad . \quad . \quad (276)$$

$$\frac{1}{L^1}=\omega^2\,\delta C_S{}^1 \quad . \quad . \quad . \quad . \quad . \quad . \quad (277)$$

where δC_S and $\delta C_S'$ are the changes in the settings of C_S and C_S', respectively. In this circuit the generator, detector, both the standard condensers, and the test coil itself, all share a common earth terminal. It will be noted that this circuit, like the type (*f*) bridge (§ 60 (*b*)) avoids the use of a variable resistance.

62. Screening of measuring circuits

At high frequencies, the problem of preventing capacitance coupling between the various parts of the measuring circuit becomes a very important problem. The introduction of metallic screens does not reduce the stray capacitances ; in fact, it will almost certainly increase them ; but these capacitances, instead of being unknown, can be arranged so that they are incorporated in the circuit and can be allowed for where necessary. As an example, let us consider the screening of one arm of a partial-substitution series resonant bridge. This is shown in Fig. 51. L_1 is the " ballast " coil, initially arranged to resonate with C_S at the test frequency. The " unknown " coil is inserted in series (at XY) and C_S readjusted to produce a new balance. Fig. 51 (*b*) shows a possible screening arrangement, consisting of three metal boxes, *a*, *b*, and *d*, placed one inside the other. In Fig. 51 (*c*), a diagram is shown of the equivalent circuit, using C_{ab} and C_{bd} to denote the capacitances between the respective pairs of screens. C_{ab} is shunted across the ballast coil and will merely effect the initial balance of the bridge. C_{bd} is shunted across the whole arm, which, at resonance, has a very low impedance and is practically unaffected by this capacitance. Another screening problem is caused by the fact that a simple bridge cannot have a common earthed lead shared by both oscillator and detector. This necessitates a balanced and screened transformer at one side of the bridge. It has already been pointed out that the screening is greatly simplified in the case of the " Twin-T " circuit, in which each component can have one side connected to a common earthed point.

63. Impedance standards

The accuracy of practically any method of impedance measurement will depend on the accuracy of calibration of one or more impedance standards, which may be in the form of coils, condensers or resistors.

(*a*) INDUCTORS

A coil used for this purpose will have losses which, since they cannot be neglected, will need to be accurately known. For this reason, and also to produce an inductance value which will not vary with frequency or current, standard inductors are usually air-cored. They should be designed so that their total resistance, over the working frequency range, is only slightly greater than the d.c. resistance of the winding. Measurements of the total resistance at a considerably higher frequency will then enable the resistance at working frequencies to be accurately calculated. The self-capacitance should be kept as small as possible, particularly when the standard coil is to be used as a series circuit element.

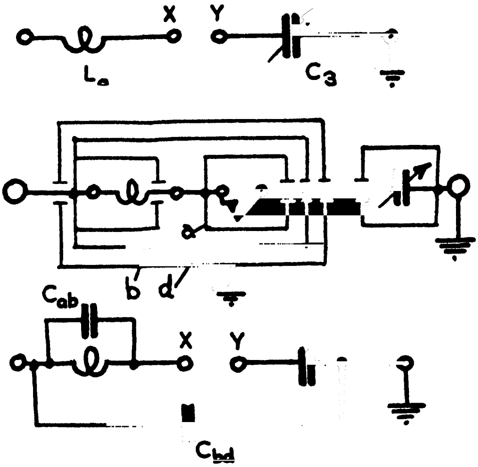


FIG. 51. Diagram illustrating the screening of a.c. measuring circuits.

(*b*) CAPACITORS

A carefully constructed air-dielectric condenser probably represents the nearest possible approach to a pure reactive impedance, since Q values of the order of 50,000 can be obtained. The dielectric losses in the small amounts of solid insulating material used in the construction of the condenser can be represented by a conductance

G, whose value for a given condenser is practically proportional to frequency, so that the ratio $\frac{G}{\omega}$ is a constant. This ratio forms a "figure of merit" which forms a useful reference for comparing the performance of various condensers. $\frac{G}{\omega}$ has the dimensions of a capacitance, and is usually expressed in micro-microfarads. A typical value for a precision laboratory-type air-condenser is 0·03 $\mu\mu$F. The Q value of a condenser is $\frac{\omega C}{G}$, so that the Q of the above condenser would be independent of frequency and equal to $\frac{C}{0{\cdot}03}$ when C is expressed in $\mu\mu$F. For example, the Q would be 33,000 when the capacitance setting was 1000 $\mu\mu$F. The Q is a maximum when the variable condenser is set at a value near its maximum capacitance ; a point worth remembering when using such a condenser for impedance measurements. At very high frequencies, the effects of the magnetic field of the condenser can no longer be neglected. This shows up as an apparent variation of capacitance with frequency, while the eddy-currents, induced by the magnetic field cutting the plates, will cause an increase in losses.

When larger capacitance values are required, condensers with solid dielectrics, such as mica, must be used. From the point of view of Q, a solid dielectric condenser is less satisfactory than an air-spaced one, partly because the absolute Q value is lower, and partly because the latter varies with frequency.

The Q frequency curve of a mica condenser passes through a maximum at a frequency whose value can be changed by changing the proportions of the condenser. The maximum Q value should be about 5000, but it may be much lower than this if the condenser is used in a frequency range far removed from that for which it was designed. At frequencies below the optimum, the fall in Q is due mainly to losses in the solid dielectric whilst, above the optimum, the losses are mostly made up of eddy-current and d.c. resistance losses in the connecting leads and in the plates themselves.

(c) RESISTORS

A wire-wound resistor behaves in the same way as an inductance coil, the only difference being in the relative magnitudes of the various effects. Eddy-currents can usually be neglected but, even if a bifilar winding is used, there will always be a small residual inductance, and there will be a distributed capacitance which can be replaced by an equivalent capacitance shunted across the terminals of the resistor. If the residual inductance and capacitance are denoted by L and C respectively, and the resistance by R, it can be shown that the variation of effective series resistance with frequency will be a minimum when R is roughly equal to $\sqrt{\frac{L}{C}}$. Under these conditions, the reactance will be negative at all frequencies. If $\sqrt{\frac{L}{C}}$ is greater than R, the reactance will be positive at low frequencies, and there will be a frequency at which the reactance will pass through zero and change sign. At this frequency, the impedance of the resistor is again a pure resistance, but its value will not be the same as that measured at low frequencies. A fixed resistance standard can be designed so that the condition $R=\sqrt{\frac{L}{C}}$ is met, although this is difficult to arrange when a variable resistor is required. These are usually of a decade type, and further residual inductances and stray capacitances are introduced by the switches and wiring. Where great accuracy is required, a variable decade *conductance* is sometimes preferable. In this, the incremental changes are brought about by switching in or out additional resistance elements in parallel instead of in series. The advantage of this method is that the impedance of each switch contact and connecting lead can be made negligible in comparison with the resistance element to which it is connected. Further, by using a variable conductance as a parallel circuit component (e.g., in a Maxwell bridge ; see Fig. 48 (*e*)), the effects of self-capacitance of the standard can be minimised.

64. Measurement of self-capacitance

The self-capacitance of a coil can be determined by making a series of measurements of the additional shunt capacitance required to resonate the coil at various frequencies. If the shunt inductance is L′ and the additional capacitance is C_1

$$\text{then } \frac{1}{L'}=(C+C_1)\,\omega^2$$

where C is the self-capacitance

$$\text{or } C_1=\frac{1}{\omega^2}\cdot\frac{1}{L'}-C.$$

Thus, assuming L′ to be independent of frequency, the graph obtained by plotting C_1 against $\frac{1}{\omega^2}$ should be a straight line, with a slope of $\frac{1}{L'}$, and an intercept on the C_1 axis which is negative but numerically equal to the required self-capacitance. In the case of a laminated coil, the effect of eddy-current shielding may show up as a deviation, of the plotted points, from a straight line. Where this occurs, the observed capacitance C_1 should be plotted against $\dfrac{1}{\omega^2\left(\dfrac{L'}{L_0}\right)}$ instead of $\dfrac{1}{\omega^2}$, the values of $\left(\dfrac{L'}{L_0}\right)$ being obtained from Fig. 18 for the frequencies concerned. If it is certain that the shunt inductance is independent of frequency, at least up to the self-resonant frequency, the self-capacitance can be found by a simpler method. This is merely to determine, by trial and error, the frequency f_r at which the coil impedance is purely resistive and then to make use of the relationship

$$C = \frac{1}{4\pi^2 f_r^2 L'}.$$

This self-resonant frequency can be found by balancing a bridge, Q-meter, etc., using another coil and choosing the frequency to be somewhere near the estimated resonant frequency of the test coil. The latter is then connected in parallel with the other and the effect on the balance condition noted. If more capacitance is required to restore the balance, the test coil is still inductive and therefore below its resonant frequency, while a decrease in tuning capacitance shows that the resonant frequency has been passed. It should be quite easy to find a frequency at which the tuning capacitance is exactly the same, whether or not the test coil is shunted across the other coil, indicating that the impedance of the test coil is now a pure resistance.*

65. Experimental analysis of coil impedance

We have shown how the self-capacitance of a coil can be measured, thus enabling the values of L and R to be calculated from the measured impedance at any frequency. Let us now consider the possibility of analysing such results, to find out something about the behaviour of the coil, with the object of finding out whether it has been correctly designed, or of predicting how it will behave at some other frequency outside the range of measurement. Neglecting eddy-current shielding, the series resistance of the coil can be expressed in the form

$$\frac{R}{L} = K + Af + Bf^2$$

$$\text{or } \frac{R - R_{dc}}{fL} = A + Bf$$

---

* The above simplified analysis assumes that the self-capacitance behaves as a single capacitance shunted across the terminals of the coils.

Thus, by determining R at various frequencies, and then plotting the quantity $\left(\frac{R-R_{dc}}{fL}\right)$ against f, a straight line should be obtained, with a slope of B, and an intercept of A. For an air-cored coil, A is zero,. and the straight line will pass through the origin. The value of A, for an iron-cored coil, includes the effect of residual and hysteresis losses and will depend on the current in the coil. These two losses can be separated by keeping the frequency constant, and varying the current. In tho Rayleigh region, the variation of resistance should be proportional to the current, and the rate of change of resistance with current can be used to calculate the hysteresis factor from the expression

$$F_h = \frac{\delta R}{\delta I} \cdot \frac{1}{fL^{\frac{3}{2}}}$$

and thence the residual factor from

$$F_c = \frac{1}{fL}\left(A - F_h I\sqrt{L}\right)$$

where A is measured at a current I.

The eddy-current factor is obtained from the expression

$$F_e = \frac{B}{f^2 L}$$

Although this method of analysis is simple to describe, considerable practical difficulties are sometimes encountered, particularly where low-loss coils are concerned. For instance, unless very accurate impedance measurements can be made, it is difficult to obtain plotted points at frequencies low enough to enable the value of the intercept A to be determined accurately. On the other hand, at higher frequencies, the graph often shows a tendency to curve upwards, making its slope uncertain. This is generally due to true dielectric loss, which introduces a term varying as f^3 (not to be confused with the loss due to moisture in the wire insulation, which varies as f^2 and changes the slope of the whole curve). Impedance terms proportional to higher orders of f can also be caused by circulating currents in a dust core, caused by capacitances between the particles ; an effect which may become apparent at radio frequencies, if the winding is allowed to approach too close to the core.

In the case of dust-cored coils, it is sometimes of interest to be able to split up the measured value of B, obtained as above, into two components, one due to the core and the other to eddy-current losses in the winding. It is not sufficient merely to measure the winding with and without the core, since the introduction of the core modifies the eddy-current losses in the winding, but the difficulty can be over-come by replacing the original winding by another, using wire with strands as thick as they can be without passing into the eddy-current-shielding region. In this way it is usually possible to increase the winding eddy-current losses until they swamp the core losses. Under these conditions, measurement with and without the core will establish the ratio by which the winding eddy-current loss is changed when the core is introduced, and once this ratio has been established for the core under consideration, it can be applied to calculate the probable eddy-current factor of any other winding on the same core.

Another important step in the analysis of coil losses is to separate out the dielectric loss in the insulating materials used in the construction of the coil, particularly the wire insulation. In cases where most of the dielectric loss is due to absorbed moisture, its effect can obviously be determined merely by placing the coil in a desiccator and measuring the fall in the apparent series resistance of the coil. The remaining dielectric loss, which should be made negligible, can only be found by an indirect method. One way is to make measurements with another winding, made of thick wire and with a small number of turns, the object being to make the dielectric loss negligible and to obtain an accurate value for the eddy-current factor, as described above. This result can then be used to measure the core loss factors accurately, since the winding eddy-current loss of the winding used can now be calculated and deducted from the total. Finally, when the various loss factors have been determined in this way, they can be used to calculate the resistance which should be obtained with the coil under test. The approximate value of the dielectric loss can then be found by subtracting this from the measured loss.

In general terms, the method of finding the magnitude of the dielectric loss in a coil with a large number of turns (since it is in such cases that it is of practical importance) merely reduces to one of making measurements with other windings with smaller numbers of turns, and therefore with smaller dielectric losses, and then using the results of these measurements to predict what the total loss of the coil under consideration would have been if there had been no dielectric loss. The remaining loss, whether actually due to the electric field or not, represents some effect which has not been taken into account and which is reducing the Q of the coil unnecessarily. It is almost invariably found to be due to the presence of some imperfect solid dielectric material in the coil, and, although the wire insulation is immediately suspect, it may be found that considerable losses are taking place in the insulating layers between the particles of a dust-core. Both these effects can be minimised by keeping the reactance of the coil as low as possible ; i.e., keeping the magnitude of the electric field, for a given current in the winding, as small as possible.

CHAPTER XIII

EXAMPLES OF COIL DESIGN

The object of the following examples is to show how the results of the preceding chapters can be used to solve some of the types of coil design problems which may arise. Where air-cored coils are concerned, the procedure is fairly straightforward, and quite accurate results can be obtained; the real difficulty is found when iron cores of any type are introduced, particularly non-toroidal dust cores of low permeability. In the latter case, exact calculation of inductance and coil losses becomes very difficult, and trial-and-error methods have to be used to a considerable extent. The main use of the theory in cases of this type is not so much to enable the performance of a coil to be calculated from first principles, but rather to analyse the results of measurements on sample coils, and to indicate the most suitable modifications to be made in order to obtain any desired change in the performance.

Example 1

> A coil is required to work at a frequency of about 5 Mc/sec. and is to have an inductance of 20 μH. The self-capacitance is to be such that the self-resonant frequency is at least 30 Mc/sec. and the Q at 5 Mc/sec. must not be less than 230.

It seems reasonable to expect that the specification might be met by means of an air-cored solenoid which, in order to keep the self-capacitance low enough, will almost certainly be restricted to a single layer. Let us start off, therefore, with a single-layer solenoid, using the optimum proportions suggested in § 44; i.e., coil length equal to coil diameter, and the wire diameter chosen to make the ratio of wire diameter to pitch equal to 0·6.

Under these conditions $Q_0 = kr\sqrt{f}$, and the value of k, from Fig. 38, is 0·10, so that, substituting $f = 5 \times 10^6$ and $Q = 230$ the coil radius becomes

$$r = \frac{Q}{k\sqrt{f}}$$

$$= \frac{230}{0{\cdot}10 \times \sqrt{5} \times 10^3}$$

$$= 1{\cdot}03 \text{ cm.}$$

The self-capacitance can now be obtained from Fig. 31, and is found to be about equal, numerically, to the coil radius for the proportions we have chosen. The self-resonant frequency for an inductance of 20 μH and a capacitance of 1 $\mu\mu$F is $\frac{10^9}{2\pi\sqrt{20}}$, which is of the order of 35 Mc/sec., so that the self-capacitance is within the required limit. The next step is to determine the number of turns required to give an inductance of 20 μH. The formula is

$$L=\frac{4\pi^2r^2N^2K_n}{l}\,.\,10^{-3}$$

(where L is in micro-henries).

The value of K_n is given by

$$K_n=\frac{1}{1+0{\cdot}9\left(\frac{r}{l}\right)-0{\cdot}02\left(\frac{r}{l}\right)^2}$$

which, for $\left(\frac{r}{l}\right)=0{\cdot}5$, is

$$K_n=\frac{1}{1+0{\cdot}45-{\cdot}005}=0{\cdot}69$$

$$L=\frac{2{\cdot}76\pi^2r^2N^2}{l}\,.\,10^{-3}$$

$$\text{or } N^2=\frac{1000Ll}{2{\cdot}76\pi^2r^2}$$

$$=\frac{1000\times20\times2{\cdot}06}{2{\cdot}76\pi^2\times1{\cdot}03^2}$$

whence $N=38$.

The pitch of the turns is then

$$\frac{2{\cdot}06}{38}=0{\cdot}054\text{ cm.}$$

and the wire diameter is $0{\cdot}054\times0{\cdot}7=0{\cdot}038$ cm.

We have thus obtained a design which appears to fulfil the requirements. There is one important point, however, which must not be overlooked. This is the fact that the wire diameter chosen is so small that the coil could not possibly be made self-supporting, and must be wound on some sort of former. This not only increases the difficulties of manufacture, but also introduces a solid insulating material into the electric field of the coil, thus increasing the losses, so that an allowance would need to be made for this in designing the coil. The magnitude of the dielectric losses would depend on the insulating material used, and on the depth of the grooves in the former, which carry the wire, and is not easy to estimate. In practice it would probably be simpler to construct a model coil and to find out, by measurement, how much the Q had been reduced by losses which had not been allowed for. Alternatively, a factor of safety can be allowed by increasing the coil dimensions by, say, 25%, and then reducing the number of turns to obtain the correct inductance.

In practice, the final design would be influenced by many factors, including cost, ease of production, etc. ; the object of the above calculation is merely to demonstrate how the results of the theoretical analysis can be applied to determine roughly the shape and size of coil which should be used for a given purpose.

Example 2

This example will illustrate the design of a multi-layer air-cored solenoidal coil, of the type which might be used for a wave filter in which the distortion produced by an iron core could not be tolerated.

> An air-cored coil is required with an inductance of 1·0 mH, the Q to be greater than 200 at all frequencies between 50 kc./sec. and 100 kc./sec. The self-capacitance must be low enough for the change of inductance with frequency to be less than 1% over this frequency range. The coil is to be in the form of a multilayer solenoid, and must have a screening-can.

To start with, the frequency of peak Q must be chosen to coincide with the geometric mean of the limiting frequencies between which the coil is to work.

$$\begin{aligned} \text{Thus } f_0 &= \sqrt{50 \cdot 10^4} \\ &= 7{\cdot}1 \times 10^4 \end{aligned}$$

The value of y (see § 40) at the lower limiting frequency is

$$\frac{5{\cdot}0}{7{\cdot}1} = 0{\cdot}71$$

so that Q must be 200 when $y = 0{\cdot}71$. The peak Q is given by

$$\begin{aligned} \hat{Q} &= \frac{Q}{2}\left(y + \frac{1}{y}\right) \\ &= \frac{200}{2}\,(0{\cdot}71 + 1{\cdot}42) \\ &= 213. \end{aligned}$$

Suppose we allow a small margin and design for a peak Q of 220, to occur at $f_0 = 7{\cdot}1 \times 10^4$ c./sec. These figures can now be used to calculate the required d.c. resistance factor K and eddy-current factor k.

Now, rearranging (193) and (194),

$$K = \frac{\pi f_0}{\hat{Q}}$$

$$k = \frac{\pi}{f_0 \hat{Q}}$$

Substituting numerical values, we get

$$K = \frac{\pi \times 7{\cdot}1 \times 10^4}{220} = 1030$$

$$k = \frac{\pi}{7{\cdot}1 \times 220 \times 10^4} = 2{\cdot}01 \times 10^{-7}$$

Before attempting to settle on the dimensions required, we must make some tentative assumption as to the shape of coil to be used.

In § 42 it was suggested that a reasonable choice is to make the length of the solenoid equal to its outside diameter and to make the ratio $\left(\frac{t}{r_1}\right)$ not greater than 0·25. Suppose we make $\left(\frac{t}{r_1}\right)$ equal to 0·25 and $\left(\frac{r_1}{l}\right)$ equal to 0·5. Fig. 10 shows that the appropriate value of γ is about 2000, so that

$$K = \gamma \cdot \frac{1}{r_1^2 \lambda}$$

$$\text{or } r_1^2 = \frac{\gamma}{K\lambda}$$

A typical value of the space factor λ for stranded wire is about 0·4. Assuming this, the outer radius of the coil is given by

$$r_1^2 = \frac{2000}{1030 \times 0{\cdot}4}$$

whence $r_1 = 2{\cdot}2$ cm.

Turning to the eddy-current factor, we find from Fig. 12 that the value of β, for the shape chosen, is about $3{\cdot}3 \times 10^{-3}$. The eddy-current factor is

$$k = \beta \lambda d^2$$

$$\text{or } d^2 = \frac{k}{\beta\lambda}.$$

Substituting, we obtain an expression for the required strand diameter of the wire

$$d^2 = \frac{2 \cdot 01 \times 10^{-7}}{3 \cdot 3 \times 10^{-3} \times 0 \cdot 4}$$

whence $d = 0 \cdot 012$ cm.

This seems quite reasonable, so we can proceed to determine the number of strands and the number of turns of wire required for an inductance of 1 mH.

The value of the constant K_n is given by (29), where r is the mean radius of the winding,

$$\text{i.e., } r = \frac{2 \cdot 2 \times 1 \cdot 75}{2} = 1 \cdot 92 \text{ cm.}$$

$$\left(\frac{r}{l}\right) = \frac{1 \cdot 92}{4 \cdot 4} = 0 \cdot 44$$

$$\left(\frac{t}{r}\right) = \frac{0 \cdot 25 \times 2 \cdot 2}{1 \cdot 92} = 0 \cdot 29$$

$$\left(\frac{t}{l}\right) = \frac{0 \cdot 25 \times 2 \cdot 2}{4 \cdot 4} = 0 \cdot 13$$

so that $K_n = 0 \cdot 63$.

The inductance formula is $L = \frac{4\pi^2 N^2 r^2 K_n}{l} \cdot 10^{-6}$

(when L is expressed in millihenries), which can be written as

$$N^2 = \frac{Ll \times 10^6}{4\pi^2 r^2 K_n}$$

$$= \frac{4 \cdot 4 \times 10^6}{4\pi^2 \times 1 \cdot 92^2 \times 0 \cdot 63}$$

or $N = 220$.

Having determined the number of turns, it is now possible to decide how many strands are required in the wire. The total cross-sectional area of the winding space is

$$lt = 4 \cdot 4 \times 2 \cdot 2 \times 0 \cdot 25$$
$$= 2 \cdot 42 \text{ sq. cm.}$$

The actual area filled by conductor is

$$2 \cdot 42\, \lambda = 0 \cdot 97 \text{ sq. cm.}$$

and the cross-section of one strand is

$$\frac{\pi d^2}{4} = 1 \cdot 21 \times 10^{-4} \text{ sq. cm.}$$

whence the total number of strands is

$$\frac{0{\cdot}97\times10^4}{1{\cdot}21}=8000$$

Thus the wire should be made up with

$$\frac{8000}{220}=36 \text{ strands.}$$

The design requirements are thus fulfilled by a solenoid with its outer diameter and length both about 4·4 cm. and with a winding about 0·55 cm. deep, made up of stranded wire composed of 36 strands of 40 S.W.G. insulated copper wire. There is still the self-capacitance to be considered, however. This is not at all easy to calculate, and its value is usually estimated from experimental data. The requirement is that the change of inductance with frequency shall not exceed 1% at the top frequency,

$$\text{i.e.,} \quad \frac{1}{1-\left(\frac{f}{f_r}\right)^2} \ngtr 1{\cdot}01$$

$$\text{or} \quad \left(\frac{f}{f_r}\right) \ngtr 0{\cdot}1.$$

This requires that the self-resonant frequency is to be at least ten times the top working frequency, and must therefore be above 1 Mc./sec. The inductance is 1 mH, so that the self-capacitance must not exceed $\frac{10^{-12}}{4\pi^2\times10^{-3}}$ farads $=25\ \mu\mu$F (approx.).

The self-capacitance of the proposed design is likely to be greater than this, and will probably lie between 50 and 100 $\mu\mu$F. If this is actually found to be so, the winding will have to be split up, or " sectionalized " into perhaps three or four sections, which are then connected in series. Another thing which can be determined experimentally far more easily than it can be calculated is the size of the screening-can. We have set a somewhat stringent requirement for the constancy of inductance with frequency, and this will mean that the screen will have to be kept well away from the coil. It will certainly need to be separated from it by a distance at least equal to the outer diameter of the coil, so that the volume of the screening-can will be at least that of a cylinder 13 cm. in diameter and 13 cm. long ; i.e., about 1700 cu. cm. Assuming that the screening-can is in the form of a cylinder with a diameter of $6r_1$ and length $(l+4r_1)$, let us try changing the ratio $\left(\frac{r^1}{l}\right)$, in order to find out whether the chosen coil proportions are the most suitable, from the point of view of screening-can volume. Suppose, for example, that the ratio $\left(\frac{r_1}{l}\right)$ is reduced to 0·2, the ratio $\left(\frac{t}{r_1}\right)$ being kept at 0·25. Proceeding as before, it is then found that $r_1 = 2{\cdot}0$ cm. and $l = 10$ cm., giving a screening-can size of 12 cm. diameter and 18 cm. long, with a volume of about 2000 cu. cm. This is larger than it was before, so we will increase the ratio $\left(\frac{r_1}{l}\right)$ to, say, 0·75. The outer radius of the coil then comes out to be about 2·4 cm. and the screening-can would be 14 cm. diameter and 13 cm. long with a volume of 2000 cu. cm. Thus the choice of a coil with equal length and outer diameter appears to give about the best possible solution to the problem. The depth of the winding has been chosen somewhat arbitrarily, and, at first sight, it might appear from Fig. 36 that the peak value of Q could be raised by making the ratio $\left(\frac{t}{r_1}\right)$ as small as possible. This is perfectly true, provided the self-capacitance is not important but, as stated in § 35, the self-capacitance increases rapidly as the winding depth is reduced. In the present case, it has been shown that sectionalizing of the winding is already necessary, in order to keep the self-capacitance within the required limit and the choice of $\left(\frac{t}{r_1}\right) = 0{\cdot}25$ would probably be found to be a reasonable one.

Example 3

In order to obtain a comparison between solenoidal and toroidal coils, we will next work out a design for an air-cored toroid to meet the same specification as that laid down in Example 2.

The size of a toroid is specified by the mean radius, and its shape by the values of the ratios $\left(\frac{a}{r_0}\right)$ and $\left(\frac{t}{a}\right)$ (see Fig. 4). In accordance with the conclusions reached in § 43, we shall start off by choosing $\left(\frac{a}{r_0}\right)$ as 0·5 and completely filling the winding space with wire, the value of $\left(\frac{t}{a}\right)$ then being about 0·3. The mean radius will be determined by the allowable d.c. resistance factor, and is calculated from the relationship

$$r_0^2 = \frac{\gamma}{K\lambda}$$

In this case

$$\gamma = 290 \left(\frac{r_0}{a}\right)^2 \left(\frac{a}{t}\right)$$
$$= 3850$$

so that, taking the space-factor λ as 0·4,

$$r_0{}^2 = \frac{3850}{1030 \times 0{\cdot}4}$$

whence $r_0 = 3{\cdot}05$ cm.

$$\text{and } a = 3{\cdot}05 \left(\frac{a}{r_0}\right) = 1{\cdot}52 \text{ cm.}$$

The strand size must now be worked out. For the optimum conditions chosen, β reaches its minimum value of about 3×10^{-3}. Thus

$$\begin{aligned} d^2 &= \frac{k}{\beta\lambda} \\ &= \frac{2 \times 10^{-7}}{3 \times 0{\cdot}4 \times 10^{-3}} \end{aligned}$$

so that $d = 0{\cdot}013$ cm.

To calculate the required number of turns, we will take the mean radius of a turn as

$$a\left[1 + \frac{1}{2}\left(\frac{t}{a}\right)\right] = 1{\cdot}75 \text{ cm.}$$

Using this value in (34), we get $L = 5{\cdot}26N^2 \times 10^{-6}$ mH.
Here, $L = 1{\cdot}0$ mH,

$$\text{so that } N^2 = \frac{10^6}{5{\cdot}26}$$

$$\text{and } N = 435.$$

The cross-sectional area of the winding is the area of the central hole in the former on which it is wound, and is thus

$$\pi\,(r_0 - a)^2 = 7{\cdot}25 \text{ sq. cm.}$$

The actual area covered by conductor is

$$7{\cdot}25\lambda = 2{\cdot}90 \text{ sq. cm.}$$

The cross-sectional area of one strand is

$$\frac{\pi d^2}{4} = 1{\cdot}33 \times 10^{-4} \text{ sq. cm.}$$

so that the total number of strands is

$$\frac{2{\cdot}90}{1{\cdot}33 \times 10^{-4}} = 2{\cdot}2 \times 10^4$$

and the number of strands in the wire is

$$\frac{2 \cdot 2 \times 10^4}{435} = 50.$$

In this example, as in the previous one, the required stranding may not be one which has been standardised by the wire manufacturer. To avoid having to order a special type of wire, it would thus be necessary to choose the nearest standard size, determine the exact value of the space factor obtainable with the chosen wire, and then make the appropriate slight modifications to the size and shape of the coil. In some cases it may be desirable to "twin" the wire; e.g., to wind the coil with two 27-strand wires in parallel, to obtain the equivalent of one 54-strand wire. This method is to be avoided if possible, however, since the wires have to be accurately twisted together before winding, in order to prevent circulating currents in the composite conductor, and this reduces the space-factor.

It is next necessary to consider the question of self-capacitance and it is here that one of the disadvantages of the toroid is found. We require that the self-capacitance shall not exceed about 25 $\mu\mu$F, but a more likely figure, for the winding chosen, is 150 to 200 $\mu\mu$F. This means that it will have to be split up into six or eight sections, and, as a result of the space taken up by the insulating partitions, which must be used to separate the sections if the full advantage of the sectionalization is to be obtained, the d.c. resistance factor will be increased, possibly by as much as 20%. This will have to be allowed for by an increase of about 10% in the dimensions of the coil, which then become:

$$r_0 = 3 \cdot 35 \text{ cm.}$$
$$a = 1 \cdot 67 \text{ cm.}$$

The total width of the wound coil will be $2(a+t) = 4 \cdot 34$ cm., and its overall diameter will be $2(r_0 + a + t)$ (approx.) $= 11 \cdot 0$ cm.

The screening can well fit closely over the coil, and may therefore take the form of a cylinder of about 11·2 cm. diameter and 4·5 cm. long, with a volume of 425 cu. cm. The corresponding solenoidal coil, designed in the last example, had a screening-can volume of about 1700 cu. cm., thus showing how the additional constructional difficulties, associated with toroidal coils, are offset by a considerable saving in space, particularly if the allowable variation of inductance with frequency is small.

Example 4

As the frequency of peak Q of an air-cored coil is reduced, the required value of K falls, and the coil has to be made bigger and bigger. In the present example it will be supposed that the space available for the coil is limited by other considerations, and we shall endeavour to determine what type of coil should be used, to make the best use of the available space.

> An air-cored coil is required to work at a frequency of 10 kc./sec. It must have a screening-can, whose volume is not to exceed 1000 cu. cm. What is the best type of coil to use and what will be the approximate Q?

The designs worked out in Examples 2 and 3 were for coils with a peak Q occurring at about 70 kc./sec. It was found that, for a peak Q of 220, even the toroid required a can with a volume of about 425 cu. cm., so that, to obtain the same peak Q at 10 kc/sec., the volume would have to be increased to

$$425 \times \left(\frac{70}{10}\right)^3 = 146{,}000 \text{ cu. cm.}$$

Since the allowed volume is only 1000 cu. cm., the peak Q would be reduced to

$$200 \times \frac{10}{70} \times \sqrt[3]{\frac{1000}{425}} = 38.$$

Under these conditions, the d.c. and eddy-current losses are equal at 10 kc./sec., and the Q at this frequency could be doubled if the eddy-current loss could be made negligible (as it would be, for example, if the strands were 46 S.W.G.). The problem then reduces to one of making γ as low as possible, irrespective of the value of β. Very little improvement can be obtained, in the case of the toroid, by increasing the depth of the winding, since the value of γ for full windings is practically independent of the ratio $\left(\frac{a}{r_0}\right)$. In Examples 2 and 3, the solenoid occupied about four times the volume of the toroid in order to obtain the same peak Q or, conversely, for the same volume, the Q of the toroid is $\sqrt[3]{4} = 1{\cdot}6$ times that of the solenoid. When only the d.c. loss is important, however, an improvement can be obtained by changing the shape of the solenoid from the optimum used in Example 2. Fig. 10 shows that γ, for a solenoid, is a minimum when $\left(\frac{t}{r_1}\right) \nless 0{\cdot}6$ and $\left(\frac{r_1}{l}\right)$ is small; i.e., for a relatively long coil with a "deep" winding. If $\left(\frac{r_1}{l}\right) = 0{\cdot}5$ and $\left(\frac{t}{r_1}\right)$ is increased to 0·6, the value of γ is reduced to about 1100 and the Q at 10 kc/sec. would be increased in the same ratio. The Q of the solenoid is now

$\frac{2000}{1100} \times \frac{1}{1{\cdot}6} = 1{\cdot}13$ times that of the toroid, for the same volume, and will therefore be $38 \times 2 \times 1{\cdot}13 = 86$.

The best solution thus appears to be a solenoid with $\left(\frac{t}{r_1}\right) = 0{\cdot}6$ and with outer diameter and length both equal to about

$$4{\cdot}4\sqrt[3]{\frac{1000}{1700}} = 3{\cdot}7 \text{ cm.}$$

The cylindrical screening-can will be 11 cm. in diameter and 11 cm. long.

Example 5

Next we will consider a typical design problem involving a dust-cored coil, showing how the most suitable size, shape and core material would be chosen. It will again be assumed that the coil is to have an inductance of 1 mH. and that its Q_0 is to be greater than 200 over the frequency range 50 to 100 kc/sec. An additional requirement is that the third harmonic e.m.f., generated by fundamental-frequency signal currents in this frequency range, is not to exceed 30 microvolts when the fundamental current is 10 mA (r.m.s.). Problems of this kind often arise in connection with multi-channel communication systems, where inductance coils for wave-filters, attenuation-equalizers, etc., must have values of Q which are high enough to produce the desired transmission characteristics and to avoid unwanted loss of signal power, while, at the same time, hysteresis distortion must be kept low to avoid intermodulation between the various channels of the system.

The given working frequency range suggests that a pot-type dust-cored coil might be suitable and, in § 33 it was shown that, for a typical shape of pot core

$$K_d = \frac{2700}{\lambda r^2 m} \qquad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (278)$$

$$F_e = \frac{e\,(m-1)}{n} \text{ (approx.)} \qquad . \quad . \quad . \quad . \quad . \qquad (279)$$

where n is about $0{\cdot}6$ (r is the outer radius of the bobbin).

The method of procedure is to determine the approximate values of K_d and F_e which are required to give the correct Q_0 frequency curve, and then to select a material which has the desired properties. It must be borne in mind that the whole process is based on so many assumptions that the result can only be regarded as a very rough estimate, intended to act as a guide to the final selection of the most suitable standard core. To start with, we will assume that the residual loss is such that $p = 1$. Fig. 40 shows that, to meet the given specification, Q_0' would then have to be about 310. Thus, neglecting the winding eddy-current loss,

$$f_0 = \sqrt{\frac{\mathrm{K_d}}{\mathrm{F_e}}} = 7 \times 10^4$$

$$\hat{\mathrm{Q}}_0{}' = \frac{\pi}{\sqrt{\mathrm{K_d F_e}}} = 310$$

whence $\mathrm{F_e} = 1{\cdot}45 \times 10^{-7}$
and $\mathrm{K_d} = 710$.

Next, let us assume the usual nominal value of 0·4 for λ, and also assume that $m = 6$, since this is a reasonable value for a dust-core of the type under consideration. (278) and (279) then give

$$\mathrm{K_d} = \frac{2700}{0{\cdot}4 \times 6 \times r^2} = \frac{1120}{r^2}$$

$$\mathrm{F_e} = 1{\cdot}4 \times 5e = 7e$$

$$\text{so that } e = \frac{1{\cdot}45}{7} \times 10^{-7} = 2 \times 10^{-8} \text{ (approx.)}$$

$$\text{and } r = \sqrt{\frac{1120}{710}} = 1{\cdot}25 \text{ cm.}$$

This value of e could be obtained, for example, with Permalloy dust of relatively large particle size and there would be no difficulty in obtaining a high enough permeability to realize the assumed inductance ratio of 6. We have now shown that, from the point of view of $\mathrm{Q_0}$, the pot core is quite satisfactory, so that there is no need to consider any other core shape. (At somewhat lower working frequencies, it might have been found that the required inductance ratio was higher than can be attained with a pot core. In that case it would probably be necessary to use a toroid.)

Now the question of hysteresis distortion must be investigated. The third harmonic e.m.f. $\mathrm{E_3}$, generated in a dust-cored coil by a given fundamental current $\mathrm{I_1}$, is given by (260) as

$$\mathrm{E_3} = 0{\cdot}6\mathrm{F_h I_1}^2 f\mathrm{L}.$$

Thus if $\mathrm{I_1} = 10^{-2}$ and $\mathrm{E_3} = 3 \times 10^{-6}$

$$\mathrm{F_h} = \frac{\mathrm{E_3}}{0{\cdot}6\mathrm{I_1}^2 f\mathrm{L}}$$

$$= \frac{3 \times 10^{-5}}{0{\cdot}6 \times 10^{-4} \times 10^5 \times 10^{-4.5}} = 0{\cdot}16.$$

(Note that the higher frequency limit has been taken because, for a given current, the harmonic e.m.f. increases with increasing frequency.)

$$\text{Now } \mathrm{F_h} = \left(\frac{m-1}{n}\right)^{\frac{3}{2}} h \sqrt{\frac{8\pi}{\mathrm{V}}} \,.\, 10^{3.5}$$

An average value of h for dust-core materials is about 5×10^{-6} and we have already assumed that $\left(\frac{m-1}{n}\right)$ is about 7.

$$\text{Thus } \sqrt{\frac{8\pi}{\mathrm{V}}} = \frac{0{\cdot}16}{7^{\frac{3}{2}} \times 5 \times 10^{-6} \times 10^{3.5}} = 0{\cdot}55$$

whence V = 84 cu. cm.

but, from § 33,

$$\mathrm{V} = 2{\cdot}25 r^3$$

whence $r = 3{\cdot}3$ cm.

This value of r, which is considerably greater than that required to obtain the given $\mathrm{Q_0}$, suggests that the dimensions of this coil will be determined by hysteresis distortion rather than by $\mathrm{Q_0}$. It should be possible to obtain a better compromise between $\mathrm{Q_0}$ and hysteresis by reducing the inductance ratio m. From (278) it is seen that a constant value of $\mathrm{K_d}$ could be maintained by keeping the product $r^2 m$ constant. If, therefore, we reduce m to 4, the required value of r to give the necessary $\mathrm{K_d}$ would be increased from 1·25 cm. to

$$1{\cdot}25 \sqrt{\frac{6}{4}} = 1{\cdot}53 \text{ cm.}$$

At the same time, the reduction of m would reduce the core volume required for the given hysteresis distortion. The hysteresis factor is proportional to $\left(\frac{m-1}{n}\right)^{\frac{3}{2}} . \mathrm{V}^{-\frac{1}{2}}$, so that the required volume will be reduced in the ratio $\left(\frac{3}{5}\right)^3$ and the corresponding value of r will therefore be $\frac{3{\cdot}3 \times 3}{5} = 1{\cdot}9$ cm.

We have now shown that the specification could probably be met with a pot core, arranged to have an inductance ratio of about 4, and having a bobbin with an outer radius of about 2 cm. At this stage, in practice, a model coil would be constructed, using the nearest standard type of core, and if this still did not quite meet the specification, the theory would be used to suggest what alterations should be made to the design. For example, a test will show whether the $\mathrm{Q_0}$ is high enough, or, possibly, whether it is hysteresis distortion which is the deciding factor. In either case the appropriate change in core size or material would then be predicted from the theory given in Chapter VI.

Example 6

Turning now to laminated cores, let us consider a typical problem; that of designing an H.T. smoothing choke, the purpose of which is to pass the d.c. component of a rectified wave but to present a high impedance to a.c. components. The choke thus has to have a specified inductance when a given d.c. current flows in the winding. The d.c. resistance of the winding is limited, either by the allowable voltage drop across the latter, or by the heat dissipated in it. As an example, it will be assumed that a choke of this type is to work on a rectified 50-cycle supply, and is to have an inductance of not less than 50 H. when carrying a d.c. current of 25 mA. The voltage drop across the choke is not to exceed 10 volts; i.e., the d.c. resistance must not be greater than 400 ohms.

Let the cross-sectional area of the core be A sq. cm. The inductance is then given by

$$L=\frac{4\pi N^2 A\mu_g}{l}\,.\,10^{-9}\text{ henries} \qquad . \quad . \quad . \quad (280)$$

where l is defined as the "magnetic length" of the core and μ_g is the apparent permeability of the core with its air-gap. (280) can be written as

$$L=4\pi\left(\frac{N^2}{l}\right)\left(\frac{A}{l^2}\right)\mu_g l^2\,.\,10^{-9}$$

$$\text{or } \mu_g l^2=\frac{L}{4\pi}\left(\frac{l}{N^2}\right)\left(\frac{l^2}{A}\right).\,10^9=k_1 \qquad . \quad . \quad . \quad (281)$$

For a given wire material and space factor, k_1 depends only on the shape of the core and not on absolute dimensions. The polarizing magnetizing force is

$$H_p=\frac{4\pi N I_p}{10l}$$

$$\text{or } H_p\sqrt{l}=\frac{4\pi I_p}{10}\sqrt{\frac{N^2}{l}}=k_2.$$

For coils of identical shape but differing in size, the d.c. resistance is proportional to $\left(\frac{N^2}{l}\right)$.

$$\text{Thus } R_{dc}=k_3\left(\frac{N^2}{l}\right)$$

where k_2 and k_3 are also shape factors depending only on the shape of the laminations and the way they are stacked to form the core. The procedure is now briefly as follows

(i) Assume a core shape and work out the value of k_3 from the shape of the corresponding bobbin. Since the d.c. resistance is fixed to start with, this gives the maximum allowable value of $\left(\frac{N^2}{l}\right)$.

(ii) Having decided on the core shape, the ratio $\left(\frac{A}{l^2}\right)$ is also known, so that k_1 and k_2 can now be worked out. The ratio $\frac{\mu_g}{H_p^4}=\frac{k_1}{k_2^4}$ can now be determined.

(iii) A straight line corresponding to this relationship can now be plotted on a set of curves such as those shown in Fig. 24, and the intersections with the various curves give the working points for the different core materials. The material which gives the largest μ_g and H_p should be chosen, since from (281), this requires the smallest value of l.

The most suitable core material and the approximate dimensions are now known.

(iv) Finally, the required air-gap ratio can be determined from curves such as those of Fig. 23.

The whole process can be repeated, if required, for several different lamination shapes, and the results then used to make a final selection of the type of core required. It will be found that, for laminations stacked to give a square cross-section for the centre limb of the core, an average value for the ratio $\left(\frac{A}{l^2}\right)$ is about 0·01. The quantity k_3, which denotes the relationship between the d.c. resistance and the core shape, is rather more variable, since it depends not only on the core but also on the shape of the bobbin and the percentage of the total winding space which is actually filled by wire. A reasonable value for a single-section bobbin, completely filled, is about $1{\cdot}4\times10^{-4}$.

Assuming this value in the above example gives

$$\left(\frac{N^2}{l}\right)=\frac{R_{dc}}{k_3}=\frac{400}{1{\cdot}4\times10^{-4}}=2{\cdot}86\times10^6$$

and, taking $\left(\frac{A}{l^2}\right)$ as 0·01,

$$\begin{aligned}k_1&=\frac{L}{4\pi}\left(\frac{l}{N^2}\right)\left(\frac{l^2}{A}\right).10^9\\&=\frac{50}{4\pi\times2{\cdot}86}\times10^5\\&=1{\cdot}39\times10^5\end{aligned}$$

$$k_2=\frac{4\pi \mathrm{I_p}}{10}\sqrt{\frac{\mathrm{N}^2}{l}}$$

$$=\frac{4\pi\times\cdot025}{10}\sqrt{2{\cdot}86\times10^6}$$

$$=53$$

so that $$\frac{\mu_g}{\mathrm{H_p}^4}=\frac{k_1}{k_2{}^4}=\frac{1{\cdot}39\times10^5}{53^4}$$

$$=1{\cdot}75\times10^{-2}.$$

Examination of Fig. 79 then shows that a possible solution is a radiometal core, working with a polarizing magnetizing force $\mathrm{H_p}$ of about 15 oersteds and with an apparent permeability of about 500. But $\mu_g l^2=1{\cdot}39\times10^5$, so that $l=16{\cdot}7$, thus fixing the approximate size of the core. It must be emphasized that the numerical values which have been quoted have been introduced mainly with the object of illustrating the method of design. They should not be taken as necessarily representing the best possible solution to this particular problem.

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